ANNUAL REPORT

OTTO DUNKEL

P. R. RIDER, Washington University

Otto Dunkel, son of Frederick W. and Elizabeth Check Dunkel, was born at Richmond, Virginia, May 25, 1869. He met a tragic death in St. Louis, Missouri, on January 15, 1951, having slipped and fallen down an embankment on the right-of-way of the Wabash Railroad. He lay at the foot of the embankment, unable to get back to the top, until death came from exposure. He was never married and is survived by a nephew and four nieces.

Dunkel was forced by circumstances to work during the years when most boys are in college, and it was not until he was twenty-four that he entered the University of Virginia in 1893. He received three degrees from that institution, M.E. in 1896, A.B. and A.M. both in 1898. During the academic year 1896–97 he was assistant in astronomy at Virginia under Professor Ormond Stone, founder of the Annals of Mathematics. From Virginia he went to Harvard, where he received an A.M. in 1899 and a Ph.D. in 1902. His dissertation, "Regular singular points of a system of homogeneous linear differential equations of the first order," was written under the direction of Professor Maxime Bôcher.

After receiving his doctor's degree he spent two years as instructor in mathematics at Wesleyan University, Hartford, Connecticut. He then went abroad for two years, studying at the University of Göttingen during 1904–05 and at the University of Paris during 1905–06. Upon returning to the United States he became an instructor at the University of Minnesota, which position he held for one year, then going to an instructorship at the University of Missouri. He was at Missouri during the period 1907–16, except for a leave of absence for the academic year 1909–10, which he spent at the University of Pisa. In 1916 he became assistant professor of mathematics at Washington University, St. Louis. He was promoted to an associate professorship in 1919 and became emeritus in 1939.

Dunkel published numerous papers in various journals, including the AMERICAN MATHEMATICAL MONTHLY, the Annals of Mathematics, the Bulletin of the American Mathematical Society, the Proceedings of the American Academy of Arts and Sciences, School Science and Mathematics, and the Washington University Studies. With E. R. Hedrick he translated into English Goursat's *Cours d'Analyse Mathématique*.

He served as an editor of the Problem Department of the MONTHLY from 1918 through 1946, being in charge of the department after 1933. He was zealous, conscientious, and very meticulous in this service. He probably worked through every solution that was submitted. If a solution was incorrect in any particular he would explain the matter to the solver and suggest a correction. Often when no solution was forthcoming to an especially difficult problem he supplied it himself. He was quite ingenious, and seemed to be equally at home in all fields of mathematics. He not only served faithfully as an editor, but remembered the Association generously in his will.

Otto Dunkel was a man of very simple tastes and lived quite frugally. He read a great deal and seemed especially fond of novels in foreign languages, notably French, Spanish, Italian, German. Although quiet and retiring he was friendly and was well liked by his colleagues. Because of his wide reading, extensive travel and a certain dry wit, he was an interesting conversationalist when in the mood. His death is a distinct loss to his friends and to mathematics.

THE INSTITUTE FOR NUMERICAL ANALYSIS OF THE NATIONAL BUREAU OF STANDARDS*

J. H. CURTISS, National Bureau of Standards

Suppose that a research worker in the course of a scientific investigation suddenly found that he had to solve 100 linear equations in 100 unknowns. Suppose further that he was afforded access to a large automatic digital computing machine, a machine with whose coding and operation he was familiar. How should he set about preparing the problem for the machine?

This situation is by no means as unrealistic as it may seem at first glance. Large sets of simultaneous equations are encountered in many data-reduction problems (those arising in geodetic triangulation problems are an example), in mathematical treatments of logistics and economics, and in the problems of engineering and applied physics. In fact, a great many of the numerical problems of applied mathematics, such as the numerical solution of integral and differential equations, can be reduced in one way or another to solving large sets of simultaneous linear equations, although in individual cases this may not be the best procedure.

The reader may say, "But how easy! After all, any high school student knows how to solve simultaneous linear equations. For example, why not write the various solutions down directly as the quotients of pairs of determinants and make the machine evaluate the determinants directly from their definition?"

A beautiful theoretical solution. Let us see how it would work in practice. Each of these determinants has 100 rows and 100 columns. By definition, the value of each determinant is the algebraic sum of $100\times99\times98\times\cdots\times3\times2\times1$ signed products, each consisting of 100 factors chosen from among the elements of the determinant so that each row and each column is represented. That is, if the definition of a determinant were used as the basis of the method, for each determinant over 10^{157} products of 100 numbers each would have to be formed and added together algebraically. If a multiplication could be done in 100

* Based on an address given to the Pacific Northwest Section of the Association at Oregon State College, Corvallis, Oregon on March 26, 1949.

microseconds on the automatic computing machine, and an addition in 10 microseconds, and if an infinitely extensive high-speed internal memory could be assumed, then each determinant would be evaluated in something over 10^{147} years of machine time! And if all the solutions to the equations are wanted, 101 of these determinants will have to be evaluated. It looks as if the machine will be busy for some time to come!

So our beautiful method turns out to be ridiculous for a problem of this size. By use of efficient methods, each determinant can be evaluated in something like 3×10^5 multiplications and a comparable number of algebraic additions. For the 101 determinants required, the straight computing time would be something like one hour. Another three to six hours must be allowed for internal logical operations like changing the instructions within the machine. However, the assumption of unlimited memory is pure nonsense; the really high-speed memories in the current crop of machines are limited to about 1,000 numbers of ten to twelve decimal digits each. To carry out the procedure, it would be necessary constantly to bring data in and out of the high-speed memory, a process which goes so slowly that, in comparison with it, computing time can be ignored. Finally, the twin problems of significant digits and round-off errors are so serious as to make it quite unlikely that any ordinary procedure suitable for five or six equations would get even a usable rough approximation to the answer to our problem with 100 equations.

What about other methods of solving the problem? As anyone knows who has gone even superficially into the matter, there is certainly no dearth of them to choose from. Dr. George E. Forsythe* has been working recently on a classification and bibliography of the methods. He distinguishes two main categories: direct methods, like the method of determinants referred to above, and iterative methods, in which the solution is arrived at by successively closer approximations. There is no room here to go into details, but perhaps a mere rapid recapitulation of some of the short names of the methods will suggest the variety of approaches available. In the direct category are the Chio and Aitken determinant methods; the Bingham, the R. Schmidt, the Lanczos,* and the Frame characteristic equation methods; the Gauss elimination method and many abbreviations thereof; the Cholesky (or square-root) and the "escalator" triangularization methods; the Gram-Schmidt, the Fox-Huskey*-Wilkinson, and the Bodewig orthogonalization methods; block elimination methods; the "below-line" device, the R. A. Fisher technique, and other special methods. In the interative category are about a dozen distinct variants of the Wittmeyer type (some of them associated with names like Gauss, Seidel, Jacobi, and von Mises), and a number of versions of the least-squares type of iterative method, including an entire sub-class first characterized as such by Rosser** and Hestenes,† and

* Member of the staff of the Institute for Numerical Analysis of the National Bureau of Standards. (This name will henceforth in these footnotes be abbreviated to NBS-INA.)

** Former Director of Research of NBS-INA.

† Member of the staff of NBS-INA.

the sub-class of gradient methods with several variants, the iterated elimination methods, and "relaxation" methods. Then too, there is a separate category of miscellaneous special methods, such as "Monte-Carlo" methods (*i.e.*, sampling methods based on probability theory), and a method of H. Levy‡ for solving equations when the solution is known to be in integers.

This catalogue is quite probably incomplete, but even as it stands, it presents a truly bewildering problem of selection. D. R. Hartree§ says [1] "It is probably the case that there is no one best method for the evaluation of the solution of a set of simultaneous linear algebraic equations, but that the best method in any particular case depends on the structure of the set of equations concerned. This is certainly true of methods for treating such equations without the use of an automatic machine, and may well still be true when such assistance is available. For example, it may be that most of the coefficients are non-zero, and are not small integers and moreover are known only approximately, either because they are derived from measurements which are subject to experimental error or because they are results of previous calculations and are subject to rounding-off errors; or it may be that each equation involves only a few of the variables, and these with coefficients which are small integers and are known to be exact. It is quite likely the most appropriate methods in the two cases will be different."

But how should the choice be made? It is clear that somehow, somewhere, guide lines must be laid down which will permit an intelligent selection of methods to be made with due regard to the peculiarities of a given system of equations and to the specifications of available computing equipment. The literature is full of expositions of special methods, illustrated in many instances by more or less suitable examples; but the over-all guide book has not yet been written. In fact, there seem to be many cases in which none of the known methods even now gives feasible solutions. Moreover, relatively little is known about how round-off errors and other errors inherent in computational work pile up in the case of many of the methods listed above when they are applied to really large sets of equations. Perhaps the only way to resolve that particular sort of question will be by means of extensive arithmetical experiments carried out on high-speed automatic machines.

Here is an important type of problem, then, which re-occurs over and over again in applied mathematics. A lot is known about it, but for the full and effective exploitation of automatic digital computing machinery, a lot more needs to be known, and what is already known needs to be pulled together in a usable form. In other words, here is an area of the science of numerical analysis in which both background research and foreground research, together with laboratory experimentation, are urgently needed to maximize the nation's return from its multimillion dollar investment in computing machinery.

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‡ Consultant of NBS-INA.

§ A former Director of Research.

It is to work on such problems that the Institute for Numerical Analysis of the National Bureau of Standards was established in 1947 with the support of the office of Naval Research of the U. S. Navy Department, and with the cooperation of the University of California. The Institute is located on the campus of the University of California at Los Angeles, and, of course, maintains close liaison with the University. Its permanent personnel complement numbers about 70 and includes some 15 scientists (mathematicians and theoretical physicists) at the post-Ph.D. level. In addition to the permanent staff, a number of well-known mathematicians have worked at the Institute on temporary appointments.

The Institute is a branch of the National Applied Mathematics Laboratories of the Bureau of Standards, which ONR played a fundamental rôle in establishing. The story of the Institute can best be told in a setting of that of the larger organization. This has been set forth at some length in a readily accessible article in Science [2] so only a brief outline is necessary here.

In 1945, a study was made of Navy Department computing requirements by the Office of Research and Invention, now ONR. Subsequently, Rear Admiral H. G. Bowen, then Chief of Naval Research, approached Dr. E. U. Condon, Director of the National Bureau of Standards, with the suggestion that ONR and NBS should jointly undertake to establish a national computing center, which would be equipped with large scale automatic machines and would engage in the development of such machines. In a later formalization of the idea, it was agreed that NBS would be solely responsible for administration of the center, and also that ONR's participation was conditional upon obtaining commitments from other agencies to support the new activity.

It might be mentioned here that ONR's decision to approach the National Bureau of Standards in connection with the operation of the proposed center was a rather natural one, in that ONR had been supporting on a temporary basis the Bureau's famous Mathematical Tables Project, originally a WPA project but financed during the war on OSRD funds. Also, in the spring of 1946, the Census Bureau had requested the NBS to construct a large automatic digital machine suitable for the preparation of census reports. Therefore, the nucleus for the center was already present at NBS early in 1946.

A year of cooperative study ensued, with both ONR and NBS personnel participating in various conferences with possible clients and applied mathematical groups all over the country. During the course of the year, the Air Materiel Command of the U. S. Air Force became interested in the proposed new organization, and made certain commitments in connection with its machine development program. This support, when added to other expressions of interest of a less substantial nature from other agencies, fulfilled the ONR requirements for outside participation. In the summer of 1947, the National Applied Mathematics Laboratories were established as a new division of the Bureau. In accordance with the conclusions reached during the cooperative study, it was organized into four branches: the Institute for Numerical Analysis

in Los Angeles; the Computation Laboratory in Washington (to carry on the work of the Mathematical Tables Project but with more emphasis on problem-solving and less on tables); the Statistical Engineering Laboratory in Washington; and the Machine Development Laboratory in Washington.

The Institute is the focal point in the mathematics division for basic research and training in the types of mathematics which are pertinent to the efficient exploitation and further development of high-speed automatic digital computing equipment. Secondary functions are to provide a computing service for Southern California and to give assistance in the formulation and analytical solution of problems in applied mathematics.

The Institute is divided into two sections, the Research section and the Mathematical Services section. The Research section has from the beginning been supported almost exclusively by ONR, although there is now some research work under way in it for the Air Comptroller of the United States Air Force, and the Bureau itself provides a small subsidy to pay for a part of the necessary experimental computing. The Mathematical Services section operates the computing service mentioned above, and in addition, conducts a modest computing machine development program. (Most of the NBS activity in the development of large-scale computing machinery is located in Washington.) The construction of a large automatic digital computing machine has recently been completed at the Institute under the direction of Dr. Harry D. Huskey. The machine, which is called the National Bureau of Standards Western Automatic Computer (SWAC for short), operates in the parallel mode and uses an electrostatic memory system, employing the so-called Williams principle. The electrostatic memory was originally put into operation with a 256 word capacity, but this is to be increased presently to 512 words. A rotating-drum magnetic memory of 10,000 word capacity is also being added. The machine was financed by the Office of Air Research of the Air Materiel Command. This Office from the start has been an important client of the computing services of the Institute.

The guiding philosophy in staffing the Institute has been that since the program is essentially mathematical, the organization should be staffed by professional mathematicians. At first glance, this idea might not seem to be very novel, but the fact is that in the field of numerical analysis, as in certain other parts of applied mathematics, there seems to be a well-established tradition of amateurism. Such a tradition has its good points, but by and large, it tends to retard an orderly and scholarly development of the discipline. There is, in fact, a popular saying among computers of the older school that nothing new has been discovered in numerical analysis since Gauss. The statement is certainly not true now, and was not quite true ten years ago, either. But there has indeed been a deficiency of professional, creative research in the theory and science of computation during the last 100 years—an era which has seen tremendous progress in many other branches of mathematics.

So a sustained effort has been made to staff the Research Section of the

Institute with competent professional mathematical research workers of established reputation. It has been found desirable to have at least one theoretical physicist and one expert in classical applied mathematics on the staff, so that advice can readily be obtained as to profitable directions in which to work.

Every care has been taken to provide attractive working conditions for the research staff. For example, most of the senior members of the research staff have private offices; experienced typists are on hand for typing mathematical manuscripts; desk computing machines are readily available for those who need them; a conscious effort is made to insulate the scientists from administrative red tape.

The Institute staff members have faculty privileges in the excellent UCLA library nearby; but a book in hand is worth two on somebody's desk in a fraternity house six blocks away, and it was soon clear that a good working collection should be built up in the Institute. The aim has been to make it preeminent in all items bearing directly on numerical analysis, both ancient and modern, and to make it quite complete in up-to-date material in mathematical analysis and applied mathematics. Secondary emphasis has been placed on theoretical physics and certain branches of pure mathematics, such as abstract algebra, although outstanding reference works in such fields are ordered. The Institute library has been fortunate in picking up back editions of the standard mathematical journals; for example, it has about all volumes of Mathematische Annalen back to vol. 1 in 1869.

The computing equipment of the Institute, in addition to the SWAC, consists of hand machines and punched-card machinery. The punched card installation includes an IBM Card Programmed Calculator (this is an electronic computing machine with a substantial internal memory), a 604 Calculating Punch, two 602A multipliers, a tabulator, and various supporting items. The research mathematicians have easy access to the machines at all times (there is no restricted area), and unless an emergency arises they usually get prompt action on any experimental computing.

Considerable emphasis is placed on the educational aspects of the Institute program. Scholars in various fields have been introduced to the facilities of the Institute through the medium of many short-term appointments of senior research workers. There have also been a number of visitors to the Institute whose travel and expenses were paid for by their own institutions. It is now well-known among scientists that they are always welcome at the Institute, and the visitors' book looks like a "Who's Who" of the Mathematical world. The remark has been heard that it is the Institute for Advanced Study in the East and the Institute for Numerical Analysis in the West that are visited by all mathematical travelers in this country.

More formal educational programs are conducted, too. The Institute collaborates with UCLA on various courses pertinent to its work. For example, an ambitious program of graduate mathematical instruction pertinent to research in modern numerical analysis is being given cooperatively by the Institute and

UCLA in the summer of 1951. Fellowships of two types are also offered: short term studentships for summer work aimed at familiarizing graduate students of mathematics and physics with the work of the Institute, and thesis fellowships lasting one or more years, designed to enable a graduate student to complete a Ph.D. thesis in a topic of interest to the Institute. A number of symposia, colloquia, and short, intensive courses on modern automatic computing machinery have also been organized from time to time.

Credit for identifying the fundamental need for the research program of the Institute should go particularly to Dr. Mina Rees, Director of the Mathematical Sciences Division of ONR. She recognized that the great activity in developing automatic digital computing machines in this country was not being adequately paralleled by theoretical investigation aimed at finding out how best to use them once they had been built, and that indeed it seemed that many of them would probably stand idle for inadmissibly long periods while mathematicians catch up with them.

The topics which have received the most attention to date by the INA research staff have been (a) the linear equation problem, mentioned in the opening paragraphs of this article, together with related matrix-inverting techniques; (b) eigenvalue problems for matrices and for continuous linear operators; and (c) numerical solutions of differential equations. Currently the last topic is one of the chief lines of research being pursued. The immediate orientation has been derived from certain difficult problems in non-linear parabolic differential equations brought into the Mathematical Services Section by one of the clients of the computing service.

In the case of linear equations, perhaps the most important task is to evaluate and study the many known methods in their relation to various types of modern automatic computing machinery. In the case of many problems of the second and third types mentioned above, it is by no means clear that appropriate methods have yet been found for really difficult cases. It is therefore interesting to try completely novel approaches.

A fruitful district in which to hunt for new methods is in probability theory. It is known that the distribution functions associated with certain random walks provide exact solutions of certain difference equations and asymptotic solutions of the related differential equations. These distribution functions can be estimated numerically by sampling procedures. Numerical techniques based in this way on probability theory are being called "Monte Carlo Methods" in computation laboratories.* Institute research workers, notably Professor Mark Kac, Dr. W. Wasow, and Dr. Robert Fortet have developed various "Monte Carlo" methods for calculating the eigenvalues of the Schrödinger equation

$$\tfrac{1}{2}\Delta u - Vu + \lambda u = 0,$$

* The reader who wishes to pursue the subject further will find a rather complete introduction and bibliography in [3].

where V is a known function and Δ is the Laplacian operator. Monte Carlo methods for solving partial differential equations and for inverting matrices have also been studied.

But this is no place, of course, for a complete catalogue of the problems worked on so far by the Institute. They are listed and described in the widely available public quarterly report entitled, "Projects and Publications of the National Applied Mathematics Laboratories." The detailed results are constantly being written up in numerous technical articles, which are sent in to the standard specialized scientific journals and the NBS Journal of Research. Over 100 of these papers have been prepared and submitted for publication since January 1, 1948.

The Institute is an example of a successful venture of the Government into the realm of *fundamental* research. It would be inappropriate, however, to bring this article to a close without saying a word or two about the immediate significance of the INA for national security.

It is well-known that automatic digital computing machinery and the associated mathematical techniques are important elements in the defense picture. The research program of the Institute is, of course, aimed quite directly at developing this national resource. More than that, the Institute maintains within its own walls a pool of equipment and a reservoir of talent in applied mathematics and computing technology which could immediately be drawn on for specific tasks in case of an acute national emergency. The organization also has built up over the last few years a substantial body of alumni and ex-staff members who are distributed all over the country, and whose contact with the Institute has made them familiar with, and interested in, modern methods of numerical analysis. Most of these persons are in academic positions, where they are able to attract and develop further talent. So although it is true that most of the research of the Institute has to date been of a very long-range, fundamental character, nevertheless the significance of the program for national security would seem to go well beyond that of the individual published results.

References

1. Calculating Instruments and Machines, D. R. Hartree, The University of Illinois Press, Urbana, 1949.

2. A Federal Program in Applied Mathematics, J. H. Curtiss, Science, 107 (1948) 257–262.

3. Sampling Methods applied to Differential and Difference Equations, J. H. Curtiss, Proceedings of Seminar on Scientific Computation held by IBM Corp. 87 (November 1949).

AN EQUILATERAL DISTANCE*

R. H. BING, University of Wisconsin

1. Introduction. A function $D(p, q)$ on the pairs of points of a topological space is called a distance for the space provided

(1) $D(p, q) \geqq 0$, the equality holding if and only if $p = q$,
(2) $D(p, q) = D(q, p)$, (symmetry),
(3) $D(p, q) \leqq D(p, r) + D(r, q)$, (triangle inequality), and
(4) $D(p, q)$ preserves limit points.

By (4) we mean that p is a limit point of the set M if and only if 0 is a limit of the set of numbers $D(p, x)$, where x is a point of M.

Examples of distance functions for the plane are

$$D_1(p, q) = \sqrt{(x_p - x_q)^2 + (y_p - y_q)^2}, \qquad D_2(p, q) = |x_p - x_q| + |y_p - y_q|,$$

and

$$D_3(p, q) = \max(|x_p - x_q|, |y_p - y_q|).$$

Furthermore, if D is any distance function, then each of the following is a distance:

$$D_4 = cD, \quad D_5 = \sqrt{D}, \quad D_6 = \min(1, D), \quad \text{and} \quad D_7 = D/(1 + D).$$

While any function $F_1(p, q)$ satisfying conditions (2), (3), and (4) in a space without isolated points also satisfies (1),

$F_2(p, q) = 2|x_p - x_q| + x_p - x_q + |y_p - y_q|$ does not satisfy (2),
$F_3(p, q) = (x_p - x_q)^2 + (y_p - y_q)^2$ does not satisfy (3), and
$F_4(p, q)$ equals 0 or 1, according as p is or is not q, does not satisfy (4).

Although general distances may be quite different from the Euclidean distance, the interiors of spheres are open sets, and spaces with distance functions have a type of uniformity that facilitates their study. Some types of distances are more like the Euclidean distance than others. For example, a distance function $D(p, q)$ is convex provided:

(5) For each pair of points p, q there is a point r different from p and q such that $D(p, q) = D(p, r) + D(r, q)$.

We call $D(p, q)$ an equilateral distance if it satisfies conditions (1), (2), (3), (4), and

(6) For each pair of points p, q there is a point r such that $D(p, q) = D(p, r) = D(r, q)$.

It may be noticed that the point r is not between p and q but forms an equilateral triple with them.

* Excerpt from an address "An equilateral distance" before the Wisconsin Section of the Association, held at Lawrence College, Appleton, Wisconsin, on May 14, 1949.

A misprint influenced the starting of this paper. The author became interested in equilateral triples as a result of looking at Gustav Beer's paper *Beweis des Satzes, dass jede im kleinen zusammenhängende Kurve convex metrisiert werden kann.* A typographical error [Fundamenta Mathematicae, vol. 31 (1938), p. 281] in the introduction of that paper caused a distance to be called convex if it satisfied (6) instead of (5). However, the "+" was intended instead of the "=" because the paper considered the assigning a convex metric to the one-dimensional image of a straight line interval under a continuous transformation. One might wonder if this continuum might be assigned a distance satisfying (6). It did not seem intuitively obvious that even an interval could be assigned such a distance. The ruler that is described below was constructed to show that it could be assigned one.

2. Equilateral triples in the plane. The ordinary distance for Euclidean 2-space (the plane) is equilateral because each pair of points lies on the vertices of an equilateral triangle. However, this is not true of Euclidean 1-space (the line).

THEOREM 1. *If M is a bounded plane set with more than 3 points, some pair of points of M does not belong to an equilateral triple in M under the metric* $D(p, q) = \sqrt{(x_p - x_q)^2 + (y_p - y_q)^2}$.

Proof. If each pair of points of M belongs to an equilateral triple, then each pair of points of $\overline{M}$ (M plus its limit points) belongs to an equilateral triple in $\overline{M}$, for suppose that p and q are points of $\overline{M}$. Let $p_1, p_2, \cdots$ and $q_1, q_2, \cdots$ be sequences of points of M converging to p and q, respectively. Suppose r_i is a point of M such that p_i, q_i, r_i is an equilateral triple. Some subsequence of $r_1, r_2, \cdots$ converges to a point r of $\overline{M}$, and p, q, r is an equilateral triple.

For some pair of points a, b of $\overline{M}$, $D(a, b)$ is the diameter of $\overline{M}$ (that is, no pair of points of $\overline{M}$ are farther apart than a and b). Suppose c is a point of $\overline{M}$ such that a, b, c is an equilateral triple.

We now show that there is a point d of $\overline{M}$ outside the triangle abc. Suppose d' is a point of $\overline{M}$ other than a, b, or c, and suppose that z is the one of a, b, c that is farthest from d'. If d' is not outside the triangle abc, then the point of $\overline{M}$ that forms an equilateral triple with z and d' is outside.

Without loss of generality, we suppose that the interval ad intersects the interval bc at f and that b and the point e of $\overline{M}$ which forms an equilateral triple with a and d are on opposite sides of the line through a and d. By the laws of cosines and sines,

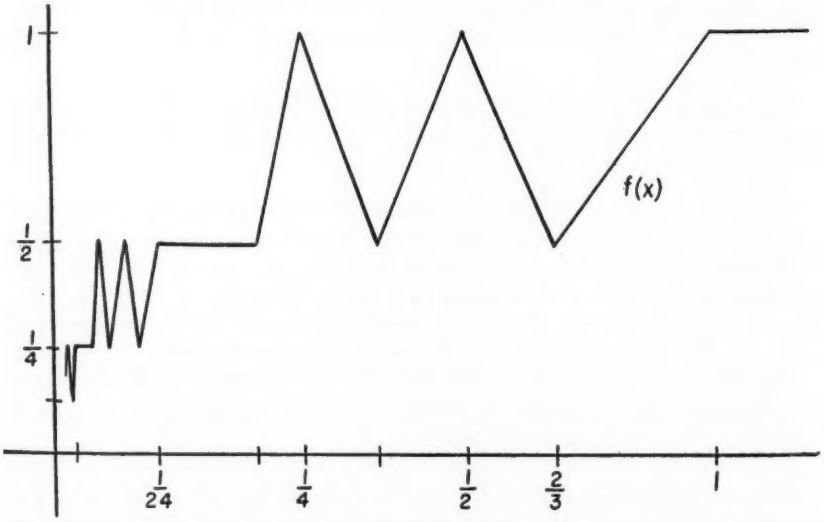
$$
\begin{aligned}
D^2(e, b) &= D^2(a, b) + D^2(a, e) - 2D(a, b)D(a, e)\cos \angle eab \\
&> D^2(a, b) + D(a, e)[D(a, f) - 2D(a, b)\cos \angle eab] \\
&= D^2(a, b) + D(a, e)D(a, b)[\sin \angle abf \\
&\quad - 2\cos \angle eab \sin \angle afb]/\sin \angle afb.
\end{aligned} \tag{7}
$$

Since $\angle eab$ and $\angle afb$ are supplementary, the expression in the bracket of the fourth member of (7) is equal to $\sin(\pi/3) - \sin(2\angle eab)$, which is positive. Hence, the assumption that each pair of points of M belongs to an equilateral triple has led us to the contradiction that e and b of $\overline{M}$ are farther apart than any two points of $\overline{M}$.

3. An equilateral ruler. The ruler portrayed in the figure that follows is the graph of a function $f(x)$ satisfying

(8) for each x, either $f(x) = f(2x)$ or $f(x) = f(x/2)$, and

$$(9) \qquad \begin{bmatrix} f(a) \leqq f(b) + f(c) \\ f(b) \leqq f(a) + f(c) \\ f(c) \leqq f(a) + f(b) \end{bmatrix} \quad \text{provided} \quad \begin{bmatrix} 0 \leqq a \leqq b + c \\ 0 \leqq b \leqq a + c \\ 0 \leqq c \leqq a + b \end{bmatrix}.$$



If $E(p, q)$ is a distance, and r_1 is the point halfway between p and q under E and r_2 is the point such that q is halfway between p and r_2, then it follows from (8) that one of the points r_1, r_2 belongs to an equilateral triple with q and p under $D(p, q) = f[E(p, q)]$. We shall now describe $f(x)$ further and show that $D(p, q) = f[E(p, q)]$ is a distance.

The graph of $f(x)$, $(1/2 \leqq x \leqq 1)$, is the sum of the interval from (1, 1) to (2/3, 1/2) and the interval from (2/3, 1/2) to (1/2, 1); $f(x) = f(2x)$, $(1/6 \leqq x \leqq 1/2)$; $f(x) = 1/2$, $(1/24 \leqq x \leqq 1/6)$; $f(x) = f(24x)/2$, $(0 < x \leqq 1/24)$; $f(0) = 0$; $f(x) = 1$, $(x \geqq 1)$. It may be noted that

$$(10) \qquad f(a)/a \geqq 12^i \cdot 9 \text{ if } 0 < a \leqq 1/24^{i+1},$$

and

(11) $$|f(c)-f(b)|/(c-b) \leqq 12^i \cdot 6 \text{ if } 1/24^{i+1} \leqq b < c.$$

It is necessary that $f(x)$ satisfy condition (9) or else $D(p, q) = f[E(p, q)]$ does not satisfy condition (3) and is not a distance. Suppose $0 < a \leqq b \leqq c$ and $1/24^{i+1} \leqq b \leqq 1/24^i$. Then $f(a) \leqq f(b) + f(c)$ because $f(a)$ is not more than twice as large as either $f(b)$ or $f(c)$. If $a \geqq 1/24^{i+1}$, then (9) holds because $c \leqq 2/24^i$ and each of the numbers $f(a), f(b), f(c)$ lies between $1/2^{i+1}$ and $1/2^i$, inclusive. If $a < 1/24^{i+1}$, then it follows from (10) and (11) that $f(a) > |f(b)-f(c)| \cdot [a/(c-b)]$. Since $a/(c-b) \geqq 1$, $f(a) > f(b)-f(c)$ and $f(a) > f(c)-f(a)$. Hence, condition (9) holds and $D(p, q)$ satisfies condition (3).

Since $f(x)$ is continuous, $f(0) = 0$, and $f(x) > 0$ if $x > 0$, $D(p, q)$ satisfies (1) and (4). It is symmetric because $E(p, q)$ is symmetric. Hence, $D(p, q)$ is a metric topological equivalent to $E(p, q)$.

To find the distance between p and q with this ruler, place it so that p falls at the origin and q is on the positive x-axis. Then $D(p, q)$ is the height of the graph above q. If this ruler is used to measure a straight line interval of length 7/4 or more, an equilateral distance is obtained. Hence,

THEOREM 2. *Each arc has an equilateral distance.*

This ruler also gives an equilateral distance to a Hilbert space and to any Euclidean space.

If n is a positive integer and bc is an interval, a ruler can be constructed in a similar fashion that will cause each pair of points of bc to belong to a collection of n points of bc such that the distance between any two of them is the distance between any other two.

A NEW INTERPOLATION FORMULA*

P. M. HUMMEL and C. L. SEEBECK, JR., University of Alabama

1. Introduction. An interpolation formula is usually a polynomial which has two or more points in common with the function (or table of functional values) in question. The interpolation formula developed here differs from those now in use in that, not only are the functional values used, but as many derivatives can be incorporated in the formula as the user desires. Usually the inclusion of one or more derivatives greatly improves the accuracy. When no derivatives are used, our formula reduces to the well known Lagrange formula. We also give an error function which can be evaluated to give both an upper and lower bound for the error.

* Presented as an address "A refinement for linear interpolation applied to approximations of the roots of equations" by C. L. Seebeck, Jr. before the Southeastern Section of the Association, held at the University of Alabama, University, Alabama, March 18, 1949.

2. The general formula. Let $f(x)$, together with its first $m+n+1$ derivatives, be continuous in an interval containing $x_0, x_1, \cdots, x_n$, where $x_i \neq x_j$. We define

$$c_i' = \frac{(m+n-i)!m!}{(m+n)!(m-i)!i!}, \qquad c_i = \frac{(m+n-i)!n!}{(m+n)!(n-i)!i!}.$$

$$S_k = f(x_k) + \sum_{i=1}^{m} c_i' f^{(i)}(x_k)(x - x_k)^i, \qquad k = 0, 1, 2, \cdots, n. \tag{1}$$

$$R_k = (-1)^n \cdot \frac{m!n!(x - x_k)^{m+n+1}}{(m+n)!(m+n+1)!} f^{(m+n+1)}(\theta_k), \tag{2}$$

where θ_k lies somewhere between x and x_k.

$$R = \sum_{i=0}^{n} \prod_{j \neq i} \frac{x_j - x}{x_j - x_i} R_i. \tag{3}$$

We are now ready to prove the

THEOREM. *Under the conditions stated*

$$f(x) = \sum_{i=0}^{n} \prod_{j \neq i} \frac{x_j - x}{x_j - x_i} S_i + R. \tag{4}$$

The interpolation formula is obtained by using the right member of (4) with R omitted, which is the error due to using the formula.

Proof. The function $f(x)$ may be expanded in a generalized Taylor Expansion* as follows

$$f(x) = f(x_k) + \sum_{i=1} \frac{(m+n-i)!}{(m+n)!} \left[{}_mC_i f^{(i)}(x_k) - (-1)^i {}_nC_i f^{(i)}(x) \right](x - x_k)^i + R_k, \tag{5}$$

where the ${}_pC_q$ are binomial coefficients and R_k is given by (2). By rearranging terms, (5) can be written

$$f(x) + \sum_{i=1} c_i f^{(i)}(x)(x_k - x)^i = S_k + R_k, \qquad k = 0, 1, \cdots, n. \tag{6}$$

These may be considered as $n+1$ linear equations in $f(x), f'(x), \cdots, f^{(n)}(x)$. The determinant of the coefficients is

$$D = \begin{vmatrix} 1 & c_1(x_0 - x) & c_2(x_0 - x)^2 \cdots c_n(x_0 - x)^n \\ 1 & c_1(x_1 - x) & c_2(x_1 - x)^2 \cdots c_n(x_1 - x)^n \\ \cdots & \cdots & \cdots \\ 1 & c_1(x_n - x) & c_2(x_n - x)^2 \cdots c_n(x_n - x)^n \end{vmatrix}.$$

This is a modified form of the Vandermonde determinant, and has the value

* A Generalization of Taylor's Expansion, P. M. Hummel and C. L. Seebeck, Jr. This MONTHLY, Vol. 56, No. 4, 1949, pp. 243–247.

$$D = \left(\prod_{i=1}^{n} c_i\right)\left(\prod_{i>j}(x_i - x_j)\right),$$

which does not vanish since the x's are distinct and the c's are different from zero. Hence we can solve the system of equations (6) for $f(x)$ by Kramer's rule, getting

$$D \cdot f(x) = \begin{vmatrix} S_0 + R_0 & c_1(x_0 - x) & c_2(x_0 - x)^2 \cdots c_n(x_0 - x)^n \\ S_1 + R_1 & c_1(x_1 - x) & c_2(x_1 - x)^2 \cdots c_n(x_1 - x)^n \\ \cdots & \cdots & \cdots \\ S_n + R_n & c_1(x_n - x) & c_2(x_n - x)^2 \cdots c_n(x_n - x)^n \end{vmatrix}.$$

We expand the above determinant according to the elements of the first column. This gives

$$D \cdot f(x) = \sum_{k=0}^{n} (-1)^k (S_k + R_k) M_k, \tag{7}$$

where

$$M_k = \begin{vmatrix} c_1(x_0 - x) & c_2(x_0 - x)^2 \cdots c_n(x_0 - x)^n \\ c_1(x_1 - x) & c_2(x_1 - x)^2 \cdots c_n(x_1 - x)^n \\ & k\text{th row missing} \\ c_1(x_n - x) & c_2(x_n - x)^2 \cdots c_n(x_n - x)^n \end{vmatrix}.$$

This is easily expanded and gives

$$M_k = \left[\prod_{i=0}^{n} c_i\right]\left[\prod_{i \neq k}(x_i - x)\right] \cdot \left[\prod_{i>j;\, i,j \neq k}(x_i - x_j)\right].$$

If we divide (7) through by D, and cancel out the factors that are common to D and M_k, the coefficient of $(S_k + R_k)$ is

$$\frac{(-1)^k \prod_{i \neq k}(x_i - x)}{\left[\prod_{i>k}(x_i - x_k)\right]\left[\prod_{k>j}(x_k - x_j)\right]}.$$

The product $\prod_{k>j}(x_k - x_j)$ can be written as $(-1)^k \prod_{j<k}(x_j - x_k)$, and the formula given in the theorem readily follows.

Bounds for the error resulting from the use of this formula may be obtained by evaluating R as given by (2) and (3). By proper choice of the θ's, both an upper and a lower bound can be established.

3. A correction for linear interpolation. When $n=m=1$, equation (4) becomes

$$f(x) = I + C + R, \tag{8}$$

where

$$I = \frac{(x_1 - x)f(x_0) + (x - x_0)f(x_1)}{x_1 - x_0} \tag{9}$$

gives the approximate value of $f(x)$ obtained by linear interpolation;

$$C = \frac{(x_1 - x)(x - x_0)}{2(x_1 - x_0)}\left[f'(x_0) - f'(x_1)\right] \tag{10}$$

is an additive correction to be applied to the interpolated value, and

$$R = \frac{(x_1 - x)(x - x_0)}{12(x_1 - x_0)}\left[(x_1 - x)^2 f'''(\theta_1) - (x - x_0)^2 f'''(\theta_0)\right], \tag{11}$$

with θ_1 between x_1 and x, and θ_0 between x_0 and x, may be used to determine bounds for the error remaining after the correction C has been added to I.

A little tedious algebra makes it possible to determine an upper bound for the absolute value of the largest possible error when x is contained in the interval (x_0, x_1). We will prove the following

THEOREM. *If $f'''(x)$ does not change sign in the interval (x_0, x_1), $R=0$ for some value of x in this interval. Moreover, if M and m are the largest and smallest values of $|f'''(x)|$ in the interval, then*

$$|R| < \left[\frac{M}{124} + \frac{M - m}{144}\right]|x_1 - x_0|^3.$$

Proof. We set $a = |f'''(\theta_0)|^{1/2}$, $b = |f'''(\theta_1)|^{1/2}$, and define

$$F(x) = \frac{(x_1 - x)(x - x_0)}{12(x_1 - x_0)}\left[(x_1 - x)^2 b^2 - (x - x_0)^2 a^2\right].$$

We assume that $a \geqq b$ since this involves only the choice of notation. By inspection it is seen that $|R| = |F(x)|$. Since the factor $(x_1-x)^2b^2-(x-x_0)^2a^2$ has opposite signs for $x=x_0$ and $x=x_1$, it must vanish for some value of x between these values, and the first part of the theorem is established.

To prove the second part of the theorem we transform $F(x)$ into a more convenient form by setting $x-x_0=(x_1-x_0)y$. Then as x varies between x_0 and x_1, y varies between 0 and 1. This transformation gives

$$F(x) = \frac{(x_1 - x_0)^3}{12} y(1 - y)\left[(1 - y)^2 b^2 - y^2 a^2\right].$$

The right-hand side is readily factored, and we have

$$F(x) = \frac{(x_1 - x_0)^3}{12} y(1 - y)[b - (b + a)y][b - (b - a)y].$$

Now clearly the factor $b-(b-a)y$ varies between b and a so that its maximum value in the interval is a. Therefore

$$|F(x)| \leqq \frac{|x_1 - x_0|^3}{12} ay(1 - y)|b - (b + a)y|.$$

The expression

$$g(y) = y(1 - y)[b - (b + a)y]$$

is a cubic polynomial which vanishes for $y=0$, $y=1$, and $y=b/(a+b)$. The maximum value of $|g(y)|$ is therefore given by one of the solutions of $g'(y)=0$, both of which are in the interval (0, 1). By the usual procedure it is found that the solutions of $g'(y)=0$ are

$$\frac{a + 2b \pm (a^2 + ab + b^2)^{1/2}}{3(a + b)}$$

and hence,

$$\max |g(y)| = \frac{1}{27(a + b)^2}[(a - b)(2a + b)(a + 2b) + 2(a^2 + ab + b^2)^{3/2}].$$

Since a function which is symmetric in two variables takes on its absolute maximum, if it has one, when these variables are equal, it can be discovered, and then proved by direct algebraic methods, that

$$\frac{(2a + b)(a + 2b)}{27(a + b)^2} \leqq \frac{1}{12}, \quad \text{and} \quad \frac{2(a^2 + ab + b^2)^{3/2}}{27(a + b)^2} \leqq \frac{a\sqrt{3}}{18} < \frac{3a}{31}.$$

Using these limits we have

$$|F(x)| \leqq \frac{|x_1 - x_0|^3}{12}\left[\frac{a^2 - ab}{12} + \frac{3a^2}{31}\right] \leqq \left[\frac{a^2}{124} + \frac{a^2 - b^2}{144}\right]|x_1 - x_0|^3,$$

and the theorem follows.

4. Illustrations. We will find the value of sin 35° by the method developed in the preceding section. We take $x_0=\pi/6$ and $x_1=\pi/4$. From (9) and (10), $I=0.569036$ and $C=0.004624$. Thus

$$\sin 35° = I + C = 0.57366 \text{ approximately.}$$

To judge the accuracy of this result we evaluate (11), using the worst possible

values for the θ's, and get $|R| \leqq 0.00014$. Actually $\sin 35° = 0.57358$, so that the error is 0.00008. It is interesting to note that if we evaluate (11) for the function $\sin x$ when the interval (x_0, x_1) is 1°, we find that $|R| \leqq 0.00000005$, so that $I+C$ gives at least seven decimal-place accuracy.

The method of the preceding section may be used for solving equations of the type $f(x)=0$. For let $y=f(x)$. The value of x which makes $y=0$, is first found by linear interpolation after which the correction C is computed and applied. Since the inverse function is encountered here,

$$C = \frac{-y_0 y_1}{2(y_1 - y_0)}\left[\frac{1}{y_0'} - \frac{1}{y_1'}\right], \quad \text{where} \quad y_i = f(x_i), \text{ and } y_i' = f'(x_i).$$

We will illustrate the procedure by solving* $\cos x = 10x$. We set $y = \cos x - 10x$ and take $x_0 = 0$, $x_1 = \pi/6$. Then $I = 0.09751$ and $C = 0.00194$, giving $x = 0.09945$. Here $|R| \leqq 0.000104$, whereas the actual error is about 0.00006. Clearly the procedure may be repeated for greater accuracy.

As a final illustration, we use the formula for $n=m=2$, and compute the value of the compound interest factor $(1.031)^{50}$. Here we take $f(x) = (1+x)^{50}$, $x_0 = 0.0275$, $x_1 = 0.03$, $x_2 = 0.035$, and $x = 0.031$. In this case equation (4) reduces to

$$(1.031)^{50} = -\frac{16}{75} S_0 + \frac{84}{75} S_1 + \frac{7}{75} S_2 + R,$$

where $S_0 = 4.22213016$, $S_1 = 4.49115519$, and $S_2 = 5.06235147$. Whence

$$(1.031)^{50} = 4.6018588 + R.$$

Using equations (2) and (3) it is found that

$$-0.00000032 \leqq R \leqq -0.00000026$$

so that we have $(1.031)^{50} = 4.6018585$, accurate to seven decimal places. When this interest factor is evaluated by linear interpolation between the rates 0.03 and 0.035, the error exceeds 2 in the second decimal place.

5. Concluding remarks. It is well known that Lagrange's interpolation formula is, except for form, identical with various other formulas using successive differences. When the x_i are equally spaced and computing machines are not available, these formulas are more convenient to use than that of Lagrange. It will be observed that formula (4) not only reduces to the Lagrange formula as a special case, but also is formally identical with it when S_i is used to replace $f(x_i)$. Consequently, with a suitable choice of origin and scale, the various formulas of the calculus of finite differences may be used to evaluate (4). For example, (4) may be written in the Newton-Gregory form

* For the usual method of solving this type of problem, see Love's Calculus, 4th Ed., page 216.

$$f(x) = S_0 + \Delta S_0 x + \frac{\Delta^2 S_0}{2!} x(x-1) + \cdots + \frac{\Delta^n S_0}{n!} x(x-1)\cdots(x-n-1) + R.$$

It should be observed that the S_k as defined by (1) need not all contain the same number of derivatives. Thus S_0 could contain m_0 derivatives, S_1 include m_1 derivatives, etc. This generalization however does not affect the form of equation (4).

Finally we wish to point out that in actual practice it is usually possible to estimate the accuracy of the formula by computing one or two of the R_i as indicated by the coefficients in equation (3).

SOME CHARACTERIZATIONS OF COMPACTNESS

V. L. KLEE, JR., University of Virginia

1. Introduction. A distance-function ρ on a topological space M will be called an allowable metric for M if the topology generated by ρ coincides with M's original topology. Space M is called metrisable if it has at least one allowable metric. (For definitions of terms used but not defined here, and for proofs of results used here, see [6; pp. 1–10, 24–27].) Niemytzki and Tychonoff [3] showed by an adaptation of the Alexandroff-Urysohn metrisation proof that a metrisable space is compact if and only if it is complete in each of its allowable metrics. Hausdorff [2] proved that each allowable metric on a closed subset of a metrisable space can be extended to an allowable metric on the whole space. By use of this fact he gave a simple proof of the above result and showed that "complete" can be replaced by "bounded." Vaughan [5] improved on these characterizations by further use of Hausdorff's extension theorem and Bing [1] gave a new proof of the latter.

In this note we establish an elementary embedding theorem and apply this to give rather simple and pictorial proofs for Vaughan's principal results as well as to establish two new characterizations of compactness which seem not to be directly obtainable from the extension theorem.

2. The embedding theorem. It is well-known (and easy to prove) that if M is a topological space then the collection $C(M)$ of all bounded single-valued real continuous functions on M, with the distance between f and g defined as $\|f-g\|$ (where $\|h\| = \sup_{x \in M} |h(x)|$), is a metric space. (By metric space we mean

a metrisable space associated with a particular one of its allowable metrics.) If (M_1, ρ_1) and (M_2, ρ_2) are metric spaces (ρ_i being an allowable metric for M_i), they are said to be isometric provided there is a bi-unique transformation $f(M_1)=M_2$ such that $\rho_1(x, y)=\rho_2(f(x), f(y))$ whenever $x, y \epsilon M_1$. The transformation f is called an isometry.

Now suppose that (M, ρ) is a metric space and for each $x \epsilon M$, let f_x be the function $\rho(x, y) \mid y \epsilon M$. For some fixed $z \epsilon M$, define $F(x)=f_x-f_z$ for all $x \epsilon M$. It is not hard to verify that $F(M) \subset C(M)$, and, in fact, that F is an isometry. This result is frequently stated as follows: "(M, ρ) can be isometrically embedded in $C(M)$." Here we shall need the following slightly different embedding theorem.

THEOREM. *Suppose that M is a metrisable space and ρ is an allowable metric for M. Then there is an allowable metric δ for M and a number $n>0$ such that*

(i) if either $\rho(x, y)<n$ or $\delta(x, y)<n$, then $\rho(x, y)=\delta(x, y)$, and (ii) (M, δ) is isometric with a subset of the boundary B of the sphere $S(\phi, n)$ in $C(M)$.

(Here ϕ is the function identically zero on M and $S(f, r)$ denotes the set of all points at distance $\leqq r$ from f.)

Proof. We may suppose without loss of generality that the diameter Δ of (M, ρ) is positive. Now define $\delta=\min(n, \rho)$, where $n=\Delta/3$ if Δ is finite and $n=1$ otherwise. It is easy to verify that δ is a distance-function and clearly (i) is satisfied so δ is allowable. Now for each $p, x \epsilon M$ let $f_p(x)=\delta(p, x)$. Consider an arbitrary $p \epsilon M$. Clearly $f_p \epsilon C(M)$. And since $n<\Delta/2$ there are points x_1 and x_2 of M such that $\rho(x_1, x_2)>2n$. But then for at least one value of i, we have $\rho(p, x_i)>n$ and hence $f_p(x_i)=\delta(p, x_i)=n$. Thus $\|f_p\|=n$ and $f_p \epsilon B$. By further use of the triangle inequality, we see that $\|f_p-f_q\|=\delta(p, q)$ whenever $p, q \epsilon M$, so $f: p \to f_p$ is an isometry on (M, δ) and the proof is complete.

Notice that the proofs in the sequel are merely the analogues for subsets of B (or of $n^{-1}B$) of those which immediately suggest themselves for subsets of the square $\{(x, y) \mid \max(|x|, |y|)=1\}$ in the plane.

3. The characterizations. A metrisable space is called *compact* if each of its infinite subsets has a limit point in the space. A metric space is called *bounded* if its diameter is finite; *totally bounded* if for each positive ϵ, it is the sum of a finite number of sets each of diameter $<\epsilon$; *complete* if each Cauchy sequence of its points converges to a point in the space; and *totally complete* if each of its bounded closed subsets is compact. It is well-known that, in each of its allowable metrics, a compact metrisable space must be complete and totally bounded. Hence in Theorems 1 to 3 below (which are the same as Theorems 1 to 3 of [5]) we need merely establish the "if" part.

THEOREM 1. *A metrisable space is compact if and only if in each allowable metric it is either complete or bounded.*

THEOREM 2. *A metrisable space is compact if and only if in each allowable metric in which it is bounded it is also either totally bounded or complete.*

THEOREM 3. *A metrisable space is compact if and only if it has an allowable metric in which it is complete, and in each such metric is either totally complete or bounded.*

Proofs of Theorems 1 to 3. Let M be a non-compact metrisable space. We see from the embedding theorem that M has a homeomorph H in the subset $U=\{f \mid \|f\|=1\}$ of the space $C(M)$ and, that if M allows a complete metric, then we may take H to be complete. Since H is non-compact, there is a denumerably infinite set $\{x_1, x_2, \cdots\}$ of points of H and a set $\{\epsilon_1, \epsilon_2, \cdots\}$ of positive numbers such that $S(x_i, \epsilon_i)$ and $S(x_j, \epsilon_j)$ are disjoint for $i \neq j$ and such that, if $y_i \epsilon S(x_i, \epsilon_i)$ for each i, then the set $\{y_1, y_2, \cdots\}$ has no limit point in H.

Now whenever $x \epsilon U$, $\epsilon > 0$, and $t \geqq -1$, let $F(x, \epsilon, t)$ be the image of $S(x, \epsilon) \cdot H$ under the transformation which takes a point y at distance η from x into the point $y+(\epsilon-\eta)\epsilon^{-1}ty$. (This is a homeomorphism which takes x into $x+tx$ and is the identity on the set of points at distance ϵ from x.)

To prove Theorem 1 we observe that the set

$$\left[H - \sum_i S(x_i, \epsilon_i)\right] + \sum_{i \text{ odd}} F(x_i, \epsilon_i, i) + \sum_{i \text{ even}} F(x_i, \epsilon_i, -1 + i^{-1})$$

is (under the metric of $C(M)$) a homeomorph of M which is neither bounded nor complete.

To prove Theorem 2 consider this same set with the metric ρ defined by $\rho(x, y) = \min(1, \|x-y\|)$.

For Theorem 3 we take the set

$$\left[H - \sum_{i \text{ odd}} S(x_i, \epsilon_i)\right] + \sum_{i \text{ odd}} F(x_i, \epsilon_i, i)$$

in the metric of $C(M)$.

A slight elaboration of this method will establish the following result, which is Lemma 1 of [4].

THEOREM 4. *A metrisable space is locally compact if and only if in each allowable metric it is locally complete.*

A proper subset of a metric space will be called a *distance-set* if to each point in the complement there is a (not necessarily unique) nearest point in the set.

THEOREM 5. *A proper subset of a metrisable space is compact if and only if in each allowable metric it is a distance-set.*

Proof. The "only if" part is well-known. Now let X be a non-compact proper subset of a metrisable space M, let H be as above, and let Y be the part of H

which corresponds to X. We may suppose that Y is closed in H, for otherwise it certainly is not a distance-set. Now since Y is non-compact there are points x_i and positive numbers $\epsilon_i (0 \leqq i < \infty)$ such that $x_i \epsilon Y$ for $i \geqq 1$, $x_0 \epsilon H - Y$, $S(x_i, \epsilon_i) \cdot S(x_j, \epsilon_j) = 0$ for $i \neq j$, and such that if $y_i \epsilon S(x_i, \epsilon_i)$ for each i then the set $\{y_1, y_2, \cdots\}$ has no limit points in $Y + S(x_0, \epsilon_0)$. Now let

$$H' = \left[H - \sum_{i=0}^{\infty} S(x_i, \epsilon_i) \right] + F(x_0, \epsilon_0, -1) + \sum_{i=1}^{\infty} F(x_i, \epsilon_i, -\tfrac{1}{2} + i^{-1}).$$

Then (in the metric of $C(M)$) H' is a homeomorph of M. Let ρ be the metric assigned to M by its natural correspondence with H' and let p be the point of M which corresponds to ϕ (in H' or to x_0 in H). Then $\rho(p, X) = \frac{1}{2}$ although $\rho(p, x) > \frac{1}{2}$ for each $x \epsilon X$. This completes the proof.

Theorem 5 acquires added interest in the light of the following remark.

THEOREM 6. *A proper subset of a metrisable space is closed if and only if in some allowable metric it is a distance-set.*

Proof. The "if" part is obvious. Now suppose conversely that X is an arbitrary closed subset of the metrisable space M and let ρ be an allowable metric for M. For each two points x and y of M define

$$\delta(x, y) = \max (\rho(x, y), 2 \left| \rho(x, X) - \rho(y, X) \right|).$$

Clearly δ is an allowable metric for M. Now consider an arbitrary point y of $M - X$. For each $x \epsilon X$ we have $\delta(y, x) \geqq 2\rho(y, X)$. However, there is a point $\bar{x} \epsilon M$ for which $\rho(y, \bar{x}) < 2\rho(y, X)$ and we then have $\delta(y, \bar{x}) = 2\rho(y, X) = \delta(y, X)$, so the proof is complete.

It is well-known that the diameter of a compact metric space must actually be attained as the distance between two of its points. Thus Theorem 7 provides another characterization of compactness.

THEOREM 7. *Each non-compact metrisable space M allows a metric ρ such that for each $p \epsilon M$, $\sup \{\rho(p, x) \mid x \epsilon M\}$ is less than the diameter of (M, ρ).*

Proof. Let H, the x_i's, and the ϵ_i's be as in the proofs of Theorems 1 to 3 above. Let g be the function identically 1 on M and let H' be the image of H under the transformation $x' = \frac{1}{2}(x + g)$. Then $H' \subset U$. Since this transformation merely halves distances, H' is a homeomorph of M. Also, the points x_i' and numbers $\epsilon_i' = \epsilon_i/2$ have the properties with respect to H' that the x_i's and ϵ_i's had with respect to H. Now let

$$H'' = \left[H' - \sum_{i} S(x_i', \epsilon_i') \right] + \sum_{i \text{ odd}} F(x_i', \epsilon_i', 1 - i^{-1}) + \sum_{i \text{ even}} F(x_i, \epsilon_i - 1 + i^{-1}).$$

H'' is a homeomorph of M, and each function in H'' has values between 0 and 2 everywhere on M. The images for odd i's of the x_i'''s have norms approaching 0, and for even i's approaching 2, so the diameter (in the metric

of $C(M)$) of H'' is 2. And since each function in H'' is bounded away from both 0 and 2 the theorem is established.

4. An additional illustration. A metrisable space is called topologically complete if it allows a metric in which it is complete. A G_δ-set is the intersection of a countable number of open sets. As an additional application of the embedding theorem, we prove the following well-known result.

THEOREM 8. *Every G_δ-set in a complete space M is topologically complete.*

Proof. Consider first the case of an open subset G of M. By the embedding theorem, there is a homeomorphism h of M into the boundary of the unit sphere of $C(M)$ such that $h(M)$ is complete. Now for each point $x \epsilon h(G)$, let $d(x)$ denote the distance from x to the set $h(M)-h(G)$. Then each point of G has a neighborhood on which d is bounded away from zero. Hence the transformation $f(x)=d(x)^{-1}x(x\epsilon G)$ is a homeomorphism taking $h(G)$ onto a complete subset of $C(M)$.

Now suppose that $G_1, G_2, \cdots$, are open subsets of M and $P=\pi_i G_i$. Each G_i allows a metric ρ_i in which it is complete and has diameter $\leqq 1$. Now for x, $y' \epsilon P$, let $\rho(x, y)=\sum_i 2^{-i}\rho_i(x, y)$. It is easily verified that ρ is an allowable metric for P. Every Cauchy sequence in P relative to ρ must also be one relative to each ρ_i, and hence converges for each ρ_i to a point of G_i. But then it must converge (relative to ρ) to a point of P and the proof is complete.

It is possible, of course, to give still further applications of the embedding theorem. For example, every locally complete metric space M can be embedded by means of a homeomorphism which preserves Cauchy sequences in a complete metric space which contains at most one point in addition to the image of M.

References

1. R. H. Bing, Extending a metric, Duke Mathematical Journal, vol. 14 (1947) pp. 511–519.

2. F. Hausdorff, Erweiterung einer Homöomorphie, Fundamenta Mathematicae, vol. 16 (1930) pp. 352–360.

3. V. Niemytzki und A. Tychonoff, Beweis des Satzes, dass ein metrisierbar Raum dann und nur dann kompact ist, wenn er in jeder Metrik vollständig ist, Fundamenta Mathematicae, vol. 12 (1928) pp. 118–120.

4. H. E. Vaughan, On locally compact metrisable spaces, Bulletin of the American Mathematical Society, vol. 43 (1937) pp. 532–535.

5. H. E. Vaughan, On the class of metrics defining a metric space, Bulletin of the American Mathematical Society, vol. 44 (1938) pp. 557–561.

6. G. T. Whyburn, Analytic Topology, American Mathematical Society Colloquium Publications, Volume XXVII, New York, 1942.

THE TWO-AREA COVERING PROBLEM*

B. M. STEWART, Michigan State College

1. Introduction. Given two regions A_1 and A_2 bounded respectively by the simple continuous closed plane curves C_1 and C_2, let us seek the boundary C of a region A which is minimal in area and which will, when properly placed, cover either A_1 or A_2. An equivalent problem is to fix A_2 in position on a coordinate system, to establish a reference point $P:(x, y)$ and a reference line $L:(\theta)$ in A_1, and to determine the position and orientation of A_1 for which $A(x, y, \theta)$, the area covered by A_1 and A_2, will be a minimum. Since, using point set notation, $A=A_1+A_2-B$, where $B=A_1A_2$ is the area covered by both A_1 and A_2, it follows that an equivalent problem is to seek the maximum of $B(x, y, \theta)$.†

In the first part of this paper we derive expressions for A_x, A_y, A_θ, under hypotheses which guarantee that A_x, A_y, A_θ are continuous, and with the aid of these expressions we characterize in a simple way the stationary values of A; namely, those positions of A_1 with respect to A_2 for which A_x, A_y, A_θ are simultaneously zero, and these values, by the usual calculus criteria, will include, of course, the minimal values of A.

In the second part of the paper we apply these results to the particular case where A_2 is a given triangle and A_1 is a circle with its center as the reference point P. Letting P^* indicate a position of P which makes A a minimum, we study the locus of P^* for circles of varying radius, and we are thus led to some interesting geometry of the triangle.

2. Partial derivatives of $A(x, y, \theta)$. Let us restrict our problem to the case where the boundary curves C_1 and C_2 can intersect in only a finite number of points, for in the sequel it will appear that otherwise A_x, A_y, A_θ may not be continuous. In counting the points of intersection let us agree to count only "crossing" points, not "touching" points. As a consequence of the Jordan theorem the number of crossing points must be even and may be designated, in order, starting from any convenient point P_1 at which C_1 enters A_2 and tracing C_1 in counterclockwise sense, as $P_i:(x_i, y_i)$, $i=1, 2, \cdots, 2n$. If the notation arises, we are to understand that $P_{i+2n}=P_i$. Let r_i indicate the length PP_i. In what follows all summations, unless otherwise indicated, run from $i=1$ to $i=2n$. With these agreements we can establish the following formulas:

* We are led to this problem by reading of the Besicovitch minimal area problem in W. W. R. Ball, Mathematical Recreations and Essays, 11th edition, New York, 1939, p. 99.

† That A actually has a minimum value is shown by a combination of facts: first, A is continuous in x, y, θ; a curve N can be found, encircling A_2 at a sufficient distance, so that, regardless of the value of θ, if P is on N, then $B=0$ and $A=A_1+A_2$; but for any fixed θ, A has a smaller value when P is located, as it can obviously be, so that $B\neq 0$, and then, by the usual continuity argument, A must attain an absolute minimum $A^*(\theta)$ for at least one $P^*(\theta)$ within N; but $A^*(\theta)$ is continuous in θ and is periodic of period 2π and hence attains an absolute minimum value for at least one θ^*, $0\leqq\theta^*\leqq 2\pi$; $A^*(\theta^*)$ is the desired minimal value of A.

If C_1 and C_2 have no crossing points, then

$$A_x = 0, \tag{1'}$$

$$A_y = 0, \tag{2'}$$

$$A_\theta = 0. \tag{3'}$$

If C_1 and C_2 have $2n$ crossing points $\{P_i\}$, then

$$A_x = -\sum (-1)^i y_i, \tag{1}$$

$$A_y = +\sum (-1)^i x_i, \tag{2}$$

$$A_\theta = +\tfrac{1}{2}\sum (-1)^i r_i^2. \tag{3}$$

Once these formulas have been established it is clear that they are independent of the "entering" point chosen as P_1, and should P_1 be chosen as a "leaving" point or should the direction of tracing C_1 be reversed, then the signs preceding the summations in (1), (2), (3) should be changed.

To establish (1) let us consider the change ΔB in $B(x, y, \theta)$ produced by giving x the increment Δx, holding y and θ constant. Consider first the case that C_1 and C_2 have just two crossing points, $U:(x_u, y_u)$ and $V:(x_v, y_v)$, with C_1 entering A_2 at U. For the simplest case, suppose that the boundary curve of A_1 is given by $x = C_1(y)$ which is single-valued from U to V, as in Figure 1.

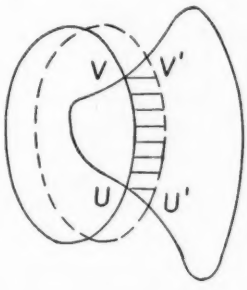


FIG. 1

Let U' and V' be the crossing points corresponding to the translated position, where the boundary curve of A_1 has the equation $x = C_1'(y) = C_1(y) + \Delta x$. Let m_u and M_u (m_v and M_v) be, respectively, the minimum and maximum values of y on C_2 between U and U' (V and V'). Since $\int \{C_1'(y) - C_1(y)\}\, dy = \Delta x \int dy$, it follows that ΔB satisfies the inequalities,

$$\Delta x(m_v - M_u) \leqq \Delta B \leqq \Delta x(M_v - m_u), \quad \text{when} \quad \Delta x > 0;$$

or the same inequalities, with both signs reversed, when $\Delta x < 0$. In either case, in the limit, $B_x = y_v - y_u$, hence $A_x = -B_x = -(y_v - y_u)$.

If $x=C_1(y)$ is not single-valued from U to V, as in Figure 2, every extra contribution to ΔB resulting from a portion of C_1 where y is decreasing is bal-

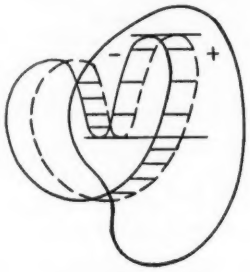


FIG. 2

anced by a contribution where y is increasing, and the final result is the same as in the simple case.

Furthermore, the same type of argument shows that a "touching" point, as in Figure 3, makes no contribution to B_x or A_x. Consequently the case where

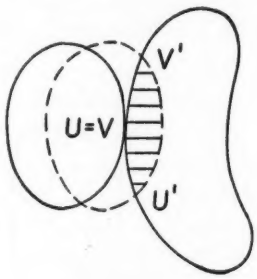


FIG. 3

there are no crossing points is one for which $B_x=0$ and hence $A_x=0$, which establishes (1').

If C_1 and C_2 have $2n$ crossing points, with C_1 entering A_2 at P_1 then for each pair of crossing points, P_{2i-1} and P_{2i}, we have the contribution $-(y_{2i}-y_{2i-1})$ to A_x, and it follows that (1) is correct.

By almost the same argument it is possible to establish (2') and (2). The change in sign, comparing (1) and (2), is not surprising if one recalls the similar situation in Green's theorem.

To establish (3) we consider the variation ΔB in $B(x, y, \theta)$ produced by giving θ the increment $\Delta\theta$, holding x and y constant. This means fixing the point P and rotating A_1 around P through the angle $\Delta\theta$, a natural situation for polar coordinates, using P as a pole and any convenient line through P as an

axis. Consider first the case that C_1 and C_2 have just two crossing points, $U:(r_u, \theta_u)$ and $V:(r_v, \theta_v)$, with C_1 entering A_2 at U. For the simplest case suppose that the boundary curve of A_1 is given by $\theta = C_1(r)$ which is single-valued from U to V as in Figure 4. Let U' and V' be the crossing points corresponding to the rotated position, where the boundary curve of A_1 has the equation

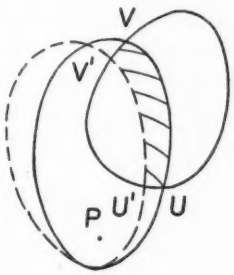


FIG. 4

$\theta = C_1'(r) = C_1(r) + \Delta\theta$. Let m_u and M_u (m_v and M_v) be, respectively, the minimum and maximum values of r on C_2 between U and U' (V and V').

Since $\int \{(C_1'(r) - C_1(r)\} r dr = \Delta\theta \int r dr$ it follows that ΔB satisfies the inequalities

$$\frac{\Delta\theta}{2}(m_v^2 - M_u^2) \leqq -\Delta B \leqq \frac{\Delta\theta}{2}(M_v^2 - m_u^2), \quad \text{when} \quad \Delta\theta > 0;$$

or the same inequalities, with both signs reversed, when $\Delta\theta < 0$. In either case, in the limit, $B_\theta = -\frac{1}{2}(r_v^2 - r_u^2)$, and $A_\theta = -B_\theta$.

If $\theta = C_1(r)$ is not single-valued from U to V, changes in ΔB resulting from a portion of C_1 where r is decreasing are balanced by changes where r is increasing, and the net result is the same as in the simple case. The same type of argument shows that touching points make no contribution to B_θ, and with this possible difficulty out of the way, it follows that when there are no crossing points then $B_\theta = 0$ and hence $A_\theta = 0$ which establishes (3'). If C_1 and C_2 have $2n$ crossing points, with C_1 entering A_2 at P_1, for each pair of crossing points, P_{2i-1} and P_{2i}, we have the contribution $+\frac{1}{2}(r_{2i}^2 - r_{2i-1}^2)$ to A_θ, and it follows that (3) is correct.

3. Stationary values of $A(x, y, \theta)$. With the formulas (1), (2), (3) in mind it is clear that the hypothesis that C_1 and C_2 can have only a finite number of points in common will guarantee the continuity of A_x, A_y, A_θ, for the continuity of the boundary curves C_1 and C_2 carries with it the continuity of x_i, y_i, r_i. But if C_1 and C_2 were allowed to have, say, in one position, a segment of a curve in common, then in this position, A_x, for example, might have a saltus, equal to the difference in the y-coordinates of the end points of the segment. Then by

the usual theorem from calculus, the minimum values of A will be included among the stationary values of A for which $A_x=0$, $A_y=0$, $A_\theta=0$, simultaneously.

According to (1'), (2'), (3') stationary values of $A(x, y, \theta)$ result when C_1 and C_2 have no crossing points. According to (1), (2), (3) stationary values of $A(x, y, \theta)$ result when C_1 and C_2 have $2n$ crossing points $\{P_i\}$, satisfying the following conditions:

$$\sum (-1)^i x_i = 0, \tag{4}$$

$$\sum (-1)^i y_i = 0, \tag{5}$$

$$\sum (-1)^i r_i^2 = 0. \tag{6}$$

A few consequences of this result are worth noting. Instead of the reference point $P:(x, y)$, suppose that the reference point $P':(x', y')$ is chosen with $x'=x+h$, $y'=y+k$. Let R_i be the length of $P'P_i$. Since

$$\begin{aligned} R_i^2 &= (x + h - x_i)^2 + (y + k - y_i)^2 \\ &= r_i^2 - 2hx_i - 2ky_i + [h^2 + 2hx + k^2 + 2ky], \end{aligned}$$

then in the following alternating sum, the bracketed terms cancel, and

$$\sum (-1)^i R_i^2 = \sum (-1)^i r_i^2 - 2h\sum (-1)^i x_i - 2k\sum (-1)^i y_i.$$

Then from (4), (5), (6) we find that for a stationary value of A, $\sum(-1)^i R_i^2=0$. This shows that the set of conditions (4), (5), (6) is independent of the choice of the reference point $P:(x, y)$.

Let T_i indicate the vector $P_{i-1}P_i$. Let L_a be a line making an angle a with the x-axis. If T_{ia} is the (signed) projection of T_i on L_a, we find $T_{ia}=(x_i-x_{i-1})\cos a+(y_i-y_{i-1})\sin a$. Then

$$\sum_{i=1}^{n} T_{2i,a} = \cos a\sum (-1)^i x_i + \sin a\sum (-1)^i y_i.$$

Hence for a stationary value of A, it follows from (4), (5) that

$$\sum_{i=1}^{n} T_{2i,a} = 0. \tag{7}$$

Similarly,

$$\sum_{i=1}^{n} T_{2i-1,a} = 0. \tag{7'}$$

Conversely, if (7) is satisfied for each of two non-parallel lines, L_a and L_b, it is easy to prove that (4) and (5) are satisfied.

In summary, if a position of A_1 with respect to A_2 determines a stationary

value of A, then the set of crossing points $\{P_i\}$ of C_1 and C_2 must either be empty, or such that the sum of the signed projections of alternate sides of the figure $\{P_i\}$ upon any line and the alternating sum of the squares of radial distances of the $\{P_i\}$ from any point must vanish.

It can be observed directly that the case of no crossing points always yields a maximum value of A when A_1 and A_2 are in such a position that $B=0$, or a minimum value of A when either A_1 or A_2 can be so placed that one completely contains the other. The case of 2 crossing points P_1, P_2 can never yield a stationary value of A, for (7) fails to hold when L_a is parallel to P_1P_2. The case of four crossing points, P_1, P_2, P_3, P_4 leads to a stationary value of A if and only if $P_1P_2P_3P_4$ forms a rectangle.* On the one hand taking L_b and L_a parallel and perpendicular, respectively, to P_1P_2 leads by (7) to the requirement that $P_1P_2P_3P_4$ be a parallelogram; but taking P to the point of bisection of the diagonals of the parallelogram, we see by (6) that the diagonals must be equal in length, hence the parallelogram must be a rectangle. Conversely, by the remark preceding the summary, when $P_1P_2P_3P_4$ is a rectangle, all of (4), (5), (6) are satisfied.

4. The circle-triangle covering problem for 0 and 4 crossing points. Let A_2 be a given triangle and let A_1 be a given circle of radius a with its center P as the reference point. Because of the perfect symmetry of A_1 it is clear that condition (6) is now trivially satisfied, and it remains to discuss the minimal values for A for the cases of 0, 4, and 6 crossing points in the light of conditions (4), (5), and (7). A position of P leading to a minimal value of A will be designated as P^*.

The case of no crossing points leads to a minimal value of A if and only if one of the figures is sufficiently large to completely contain the other. If the triangle is sufficiently large to contain the circle we must have $a \geqq r$, where r is the radius of the inscribed circle, and we can then locate P^* at any point within a triangle which is within the given triangle and which has its sides parallel to, and at a distance a from, the sides of the given triangle. To discuss the situation when the circle contains the triangle we need to make a division into cases. If the triangle is acute, this circumstance arises when $a \geqq R$, where R is the radius of the circumscribed circle, and we can then locate P^* at any point which is common to the three circles, each having a vertex of the given triangle as a center and each of radius a. If the triangle is obtuse, the circle can contain the triangle if $a \geqq m$, where m is one half the length of the longest side, with P^* located as in the previous case.

The case of two crossing points is ruled out by the discussion concluding §3. The same discussion shows that in the case of four crossing points, $P_1P_2P_3P_4$ must form a rectangle. This is impossible unless two of the points, say P_2 and P_3, are on one side, with P_2 the projection of P_1, and P_3 the projection of P_4,

* Compare this result with C. S. Ogilvy's remark in solving E 814, this MONTHLY, vol. 56, 1949, p. 37.

and P_1P_4 parallel to P_2P_3. P^* must be at the center of the rectangle with $P^*P_1 = P^*P_4 = a$. Furthermore, since there are to be just four crossing points, we must have $P^*V_1 \leqq a$, where V_1 is the vertex of A_1 opposite the edge containing P_2 and P_3. This requires the angle at V_1 to be obtuse, hence this case of four crossing points can arise only for an obtuse triangle. The locus of P^* (for circles of varying radius) considered only as the center of the inscribed rectangle $P_1P_2P_3P_4$ is the straight line segment joining the midpoint M_1 of V_2V_3 with the midpoint H_1 of the altitude drawn from V_1. But if, see Figure 5, we note that triangle $P_1P_4V_1$ is homothetic to the given triangle $V_2V_3V_1$, we see that the circumcenter of $P_1P_4V_1$ is on the line joining V_1 to O, the circumcenter of $V_2V_3V_1$, and then the restriction $P^*V_1 \leqq a$ means that we have this case of four crossing points only when a is in the range $p \leqq a < m$, where $p = O_1V_1$

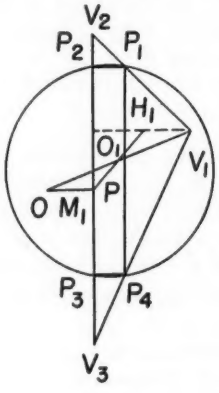


FIG. 5

and O_1 is the intersection of the lines M_1H_1 and OV_1. In case the triangle $V_1V_2V_3$ is isosceles, O_1 is not defined by this construction. An alternative definition of O_1 covering this exception is noted in the next section.

5. The circle-triangle covering problem for 6 crossing points. The remaining discussion will show that there are no other stationary values of A for $a \geqq R$ or $a \leqq r$ for an acute triangle, or for $a \geqq p$ or $a \leqq r$ for an obtuse triangle, and this leaves only the following, but most interesting, cases that there should be six crossing points with $r < a < R$ for an acute triangle and $r < a < p$ for an obtuse triangle. Before proceeding further we should note that the case of the right triangle is easily included in the discussion of either of the above cases, for it is the in-between case for which $p = m = R$.

Let (d_1, d_2, d_3) indicate the signed distances of a point $P:(x, y)$ from the respective sides V_2V_3, V_3V_1, V_1V_2 of the given triangle, with the agreement that d_1, d_2, d_3 are all positive for points interior to the triangle. Let (x_1, x_2, x_3) be the trilinear coordinates of P, with $x_1 = kd_1$, $x_2 = kd_2$, $x_3 = kd_3$, $k \neq 0$. Let

s_1, s_2, s_3 and c_1, c_2, c_3 indicate the sines and cosines, respectively, of the interior angles of the triangle at V_1, V_2, V_3, respectively. If we suppose that the circle of radius a and center P intersects the triangle in six crossing points, necessarily two on a side, and if, for the moment, we neglect the fact that these two points are supposed to have real coordinates and be on the side, not the side extended, then condition (7), applied to lines L_1, L_2, L_3 which are the respective altitudes drawn from V_1, V_2, V_3, gives the following conditions for $P:(x, y)$ to be P^*—the second set of conditions being a more easily identified form of the first set:

$$s_3\sqrt{a^2 - d_2^2} = s_2\sqrt{a^2 - d_3^2}, \tag{8.1}$$

$$s_1\sqrt{a^2 - d_3^2} = s_3\sqrt{a^2 - d_1^2}, \tag{8.2}$$

$$s_2\sqrt{a^2 - d^2} = s_1\sqrt{a^2 - d^2}; \tag{8.3}$$

$$s_3^2d_2^2 - s_2^2d_3^2 = (s_3^2 - s_2^2)a^2, \tag{9.1}$$

$$s_1^2d_3^2 - s_3^2d_1^2 = (s_1^2 - s_3^2)a^2, \tag{9.2}$$

$$s_2^2d_1^2 - s_1^2d_2^2 = (s_2^2 - s_1^2)a^2. \tag{9.3}$$

Of course the third equation in each set is redundant. To find the locus of P^* we eliminate a by adding the equations (9.1), (9.2), (9.3) and we obtain, after changing to trilinear coordinates,

$$(s_2^2 - s_3^2)x_1^2 + (s_3^2 - s_1^2)x_2^2 + (s_1^2 - s_2^2)x_3^2 = 0. \tag{10}$$

Equation (10) in general represents a conic E with respect to which the given triangle is self-polar. If the triangle is isosceles, but not equilateral, E is determined by (10) but is degenerate. If the triangle is equilateral, (10) is meaningless, but equations (8) show that the locus of P^* is a single point, the incenter. It is easily checked that E passes through incenter I and the excenters I_1, I_2, I_3 of triangle V_1, V_2, V_3, for these points have the following trilinear coordinates: $I:(1, 1, 1)$; $I_1:(-1, 1, 1)$; $I_2:(1, -1, 1)$; $I_3:(1, 1, -1)$. But I, I_1, I_2, I_3 form an orthocentric system, and every conic through an orthocentric system is known to be an equilateral hyperbola with its center Q on the circumcircle of the related diagonal triangle,* which in this case is $V_1V_2V_3$; hence this description fits the conic E. Furthermore, it is easily seen that the symmedian point S (which is the intersection of the symmedian lines, which are isogonal conjugates of the familiar median lines) and the exsymmedian points S_1, S_2, S_3 (which are the vertices of a triangle formed by the tangent lines drawn to the circumcircle at V_1, V_2, V_3) are points of E because these points have the following trilinear coordinates: $S:(s_1, s_2, s_3)$; $S_1:(-s_1, s_2, s_3)$; $S_2:(s_1, -s_2, s_3)$; $S_3:(s_1, s_2, -s_3)$. From the well-known identity $c_i^2 = 1 - s_i^2$, it then follows readily that the points $O:(c_1, c_2, c_3)$; $O_1:(-c_1, c_2, c_3)$; $O_2:(c_1, -c_2, c_3)$; $O_3:(c_1, c_2, -c_3)$ are on E; but O, the circumcenter of $V_1V_2V_3$ and O_1, O_2, O_3, may be determined

* For example, see T. F. Holgate, Projective Pure Geometry, New York, 1930, p. 207.

as intersections of lines, one through each vertex and harmonic conjugate to the circumradial line drawn to that vertex with respect to the sides of the triangle meeting at that vertex; O_1 is exactly the O_1 of the preceding section—namely, the intersection of M_1H_1 and OV_1, when this construction is valid; and similarly, for O_2 and O_3.

The restriction that the six crossing points neither have non-real coordinates nor be on the extended sides of triangle $V_1V_2V_3$ is readily investigated, for E is determined in terms of the parameter a by functions that are continuous. For an acute triangle the restriction is that P^* be on that segment of E between I and O for which $r<a<R$. For an obtuse triangle the restriction is that P^* be on that segment of E between I and O_1 for which $r<a<p$.

This completes the discussion of the locus of P^*, except for a few remarks concerning the hyperbola E and the meaning of equations (9).

Equation (9.1) represents an hyperbola E_{a1} with its center at V_1 and with the symmedian and exsymmedian through V_1 as its asymptotes. Four points on E_{a1} which are easily constructed are those for which $d_2=\pm a$, $d_3=\pm a$. In the same manner (9.2) and (9.3) represent hyperbolas E_{a2} and E_{a3}, respectively. The set of four hyperbolas E, E_{a1}, E_{a2}, E_{a3} intersect not only at $P_a^*:(x_1, x_2, x_3)$ but also at $P_{a1}:(-x_1, x_2, x_3)$; $P_{a2}:(x_1, -x_2, x_3)$; $P_{a3}:(x_1, x_2, -x_3)$. Since the construction of the points of intersection of two conics is, in general, a fourth order problem, it will not be possible, in general, to construct P_a^* in the six point case, with ruler and compass alone.

The center Q of E is the pole, with respect to E, of the line at infinity whose equation is $s_1x_2+s_2x_2+s_3x_3=0$. Hence we find that Q is the point $(2s_1^2c_1-s_2s_3, 2s_2^2c_2-s_3s_1, 2s_3^2c_3-s_1s_2)$. But the point W (which is the center of the Brocard circle) midway between S and O has the coordinates $(s_1^2c_1+s_2s_3, s_2^2c_2+s_3s_1, s_3^2c_3+s_1s_2)$ and the median point M is given by (s_2s_3, s_3s_1, s_1s_2). Hence $Q=2W-3M$ is on the line WM. Furthermore, it can be shown by reverting to metric coordinates that M is always between W and Q. Thus a neat construction for Q is afforded by finding the intersection of the circumcircle of the given triangle with the ray drawn from W through M.

There does not seem to be, in general, any simple construction for the asymptotes of E, but in the particular case of a right triangle the following construction is of interest. Let V_1 be the vertex opposite the hypotenuse. With V_2 as a center construct a circle s_2 with radius v_2, where v_2 is the length of the side opposite V_2. Let S_2 intersect the line V_1V_2 in the points X_2 and Y_2 with V_2 between X_2 and V_1. With V_3 as a center construct a circle S_3 with radius v_3 where v_3 is the length of the side opposite V_3. Let S_3 intersect V_1V_3 in the points X_3 and Y_3 with V_3 between X_3 and V_1. Let S_2 and S_3 intersect in the points Q and Q' with Q on the same side of V_2V_3 as V_1. Then the hyperbola E has Q as center, QV_1 and QQ' as asymptotes, and QX_2Y_3 and QX_3Y_2 as axes.

By way of summary, we describe an interesting model obtained from the circle-triangle problem using (x, y, a) rectangular coordinates and plotting the locus of $P^*:(x, y)$ against a. For values $0\leqq a\leqq r$ the locus is a solid tetrahedron

with the points V_1, V_2, V_3 in the $a=0$ plane as a base and with (x_I, y_I, r) as a vertex. For an acute triangle and for values $r<a<R$ the locus is a curved filament, extending from (x_I, y_I, r) to (x_0, y_0, R), the projection of this filament being a segment of the equilateral hyperbola E. For an obtuse triangle and for values $r<a<p$ the locus is a curved filament, extending from (x_I, y_I, r) to (x_{0_1}, y_{0_1}, p), the projection of this filament being a segment of E, but for values $p \leqq a \leqq m$, the locus is a straight line filament extending from (x_{0_1}, y_{0_1}, p) to (x_{M_1}, y_{M_1}, m). The remainder of the locus, in each case, is the solid common to three right-angled circular cones with vertical axes located at V_1, V_2, V_3, respectively. The model for an obtuse triangle is shown in Figure 6.

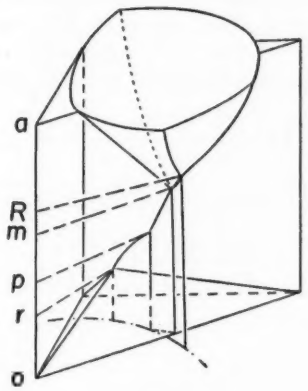


FIG. 6

6. Generalizations. The results given in equations (1)–(7) carry over to the case that A_1 and A_2 are finitely multiply-connected, providing that in setting up these equations a consistent direction of traversing C_1 is maintained with the interior of A_1 always on the left. The locus of P^* for a circle and polygons, other than a triangle, may be considered, but a cursory examination reveals here a situation too complicated with special cases to be of much interest. The problem of the minimal covering for three or more areas is a challenging one involving many variables. Finally, the generalizations to more dimensions may be considered, in particular, the three-dimensional problem. Here topological considerations enter as to the nature of the surfaces bounding the solids and the nature and orientation of the curves of intersection. Analogues to (7) are apparent, but analogues to (6) seem more difficult.

MATHEMATICAL NOTES

EDITED BY F. A. FICKEN, *New York University*

Material for this department should be sent to F. A. Ficken, Institute for Mathematics and Mechanics, New York University, 45 Fourth Ave., New York 3, New York.

ITERATED SUMS OF POWERS OF THE BINOMIAL COEFFICIENTS*

E. B. SHANKS, Vanderbilt University

Since James Bernoulli's formulas for "Summa Potestatum" were given in *Ars Conjectandi*, considerable interest has been manifested by mathematicians in this problem. It is our purpose to give a two parameter generalization of these formulas by giving a formula (equation (6) below) for the pth iterated sum of the kth powers of the $(i+1)$th binomial coefficient which we denote by $S_i^k(n, p)$.

For x any integer (positive, negative, or zero) and y a positive integer, let

$$\binom{x}{y} = \frac{x(x-1)\cdots(x-y+1)}{y!}. \tag{1}$$

It is well known (and can be easily proved by a simple induction) that

$$\sum_{i=1}^{n}\binom{i}{y} = \binom{n+1}{y+1}. \tag{2}$$

This latter equation is valid for all positive integers n and y, where the symbols are defined by (1).

LEMMA 1. *For n any integer, k and i any positive integers,*

$$\binom{n}{i}^k = \binom{n+ik-i}{ik} + A_2\binom{n+ik-i-1}{ik} + \cdots + A_{ik-i}\binom{n+1}{ik} + \binom{n}{ik}, \tag{3}$$

where A_j for $j=2, 3, \cdots, ik-i$ is independent of n.

Proof. Each side of (3) is a polynomial in n of degree at most ik so that the lemma will be proved if it is shown that these polynomials are equal for $ik+1$ distinct values of n. For $n=-1, 0, 1, 2, \cdots, i$ the polynomials are equal irrespective of the values of the A_j as may be verified by substitution and the use of the definition (1). For $n=i+1, i+2, \cdots, ik-1$ the values of $A_2, A_3, \cdots, A_{ik-i}$ are determined successively in a unique manner as positive integers independent of n, again by substitution and the use of (1). For these uniquely determined values of the A_j, the polynomials are equal for the $ik+1$ distinct values $-1, 0, 1, 2, \cdots, ik-1$ and the lemma follows.

* Presented to the Southeastern Section of the Association at Gainesville, Florida, April 8, 1950.

Note that equation (3) is a two parameter generalization of the fact that any sum of two consecutive triangular numbers is a perfect square.

LEMMA 2. *In equation* (3), *for* $j=2, 3, \cdots, ik-i$, *the coefficients* A_j *have the property that*

$$A_j = A_{ik-i-j+2} \tag{4}$$

and are given explicitly by

$$A_j = \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 & \binom{i}{i}^k \\ \binom{ik+1}{ik} & 1 & 0 & \cdots & 0 & \binom{i+1}{i}^k \\ \binom{ik+2}{ik} & \binom{ik+1}{ik} & 1 & \cdots & 0 & \binom{i+2}{i}^k \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ \binom{ik+j-1}{ik} & \binom{ik+j-2}{ik} & \binom{ik+j-3}{ik} & \cdots & \binom{ik+1}{ik} & \binom{i+j-1}{i}^k \end{vmatrix}. \tag{5}$$

Proof. In (3) let n take successively the values $i+j-1$ and $-j$, and we get, respectively,

$$\binom{i+j-1}{i}^k = \binom{ik+j-1}{ik} + A_2\binom{ik+j-2}{ik} + \cdots + A_{j-1}\binom{ik+1}{ik} + A_j,$$

$$\binom{i+j-1}{i}^k = \binom{ik+j-1}{ik} + A_{ik-i}\binom{ik+j-2}{ik} + \cdots + A_{ik-i-j+3}\binom{ik+1}{ik} + A_{ik-i-j+2},$$

where we have used the identity $\binom{x}{y} = (-1)^y\binom{y-x-1}{y}$ in deducing the second equation. Now let $j=2, 3, \cdots, ik-i$ in succession in these equations, and we get $A_j = A_{ik-i-j+2}$ as stated in the lemma. Consider $A_1=1$, the coefficient of $\binom{ik+j-1}{ik}$ in the first equation, and let j assume the values $1, 2, \cdots, j$. The resulting system of j equations in j unknowns has the unique solution (5) by Cramer's rule, the determinant of coefficients having the value $+1$.

Let $S_i^k(n, o) = \binom{n}{i}^k$ and

$$S_i^k(n, p) = S_i^k(1, p-1) + S_i^k(2, p-1) + \cdots + S_i^k(n, p-1),$$

where i, k, n and p are positive integers. We then may state the following

THEOREM. *It is identically true that*

$$S_i^k(n, p) = \binom{n + ik - i + p}{ik + p} + A_2\binom{n + ik - i + p - 1}{ik + p} + \cdots + A_{ik-i}\binom{n + p + 1}{ik + p} + \binom{n + p}{ik + p}, \tag{6}$$

where the A_j are determined by (4) *and* (5).

Proof. This follows immediately from Lemmas 1 and 2 using (2) in succession p times, starting with the identity (3).

It is an immediate consequence that $S_i^k(n, p)$ is a polynomial in n of degree $ik+p$.

If we set $A_1 = A_{ik-i+1} = 1$ and equate the coefficients of the highest power of n in equation (3), we obtain

$$\sum_{j=1}^{ik-i+1} A_j = \frac{(ik)!}{(i!)^k}.$$

In the formula for $S_i^k(n, p)$, Bernoulli's results are given by taking $i = p = 1$. The case $i = p = 1$ was discussed also by E. E. Witmer in an article, "The Sums of Powers of Integers," this MONTHLY, Vol. 42, No. 9, Part I, November, 1935, pp. 540–548. The case $i = 1$, $k = 2$ was the subject of Problem 22, *Mathematics Magazine*, Vol. XXII, No. 1, p. 51, while $i = 1$, $k = 3, 4, 5$ was the subject of Problem 4380, this MONTHLY, Vol. 57, No. 2, February, 1950, where the statement is made, "If possible, determine the form for general k." The case $i = 2$, $k = 1, 2, 3, 4, 5$ was investigated by P. A. Piza and presented at the meeting of the Southeastern section of the Association. His paper included a method for determining the formula in a recursive manner for $i = 2$ and k any positive integer. For a review of the previous work in this field the reader is referred to the references given in Witmer's article.

If in equation (3) we take $i = 1$, we get

$$n^k = \binom{n + k - 1}{k} + A_2\binom{n + k - 2}{k} + \cdots + A_{k-1}\binom{n + 1}{k} + \binom{n}{k},$$

which gives a formula for the kth power of any integer n in terms of a linear combination of k successive $(k+1)$th binomial coefficients, beginning with the $(k+1)$th of order n (reversing the order given in the equation above). The A_j are positive integers depending only on k. For example,

$$n^3 = \binom{n}{3} + 4\binom{n + 1}{3} + \binom{n + 2}{3},$$

$$n^4 = \binom{n}{4} + 11\binom{n + 1}{4} + 11\binom{n + 2}{4} + \binom{n + 3}{4}.$$

From a set of equations such as the above or by calculating the A_j, we obtain the following table:

$$S_1^1(n, 1) = \binom{n+1}{2}$$

$$S_1^2(n, 1) = \binom{n+2}{3} + \binom{n+1}{3}$$

$$S_1^3(n, 1) = \binom{n+3}{4} + 4\binom{n+2}{4} + \binom{n+1}{4}$$

$$S_1^4(n, 1) = \binom{n+4}{5} + 11\binom{n+3}{5} + 11\binom{n+2}{5} + \binom{n+1}{5}$$

. . . . . .

$$S_1^{10}(n, 1) = \binom{n+10}{11} + 1013\binom{n+9}{11} + 47840\binom{n+8}{11} + 455192\binom{n+7}{11}$$
$$+ 1310354\binom{n+6}{11} + 1310354\binom{n+5}{11} + 455192\binom{n+4}{11}$$
$$+ 47840\binom{n+3}{11} + 1013\binom{n+2}{11} + \binom{n+1}{11}.$$

THE DISTANCE SET FOR THE CANTOR DISCONTINUUM

W. R. Utz, University of Missouri

In this note there is given what seems to be a new and simple proof of a slight generalization of the known theorem (cf. J. F. Randolph, *Distances between points of the Cantor set*, this Monthly, vol. 47 (1940), pp. 549–551; *Some properties of sets of the Cantor type*, Journal of the London Mathematical Society, vol. 16 (1941), pp. 38–42) that the distance set for the Cantor middle third set based on the unit interval is the set of all real numbers c such that $0 \leqq c \leqq 1$.

Theorem. *Given any real numbers c and m such that $0 \leqq c \leqq 1$ and $\frac{1}{3} \leqq |m| \leqq 3$, there exist numbers x and y representing points on the Cantor middle third set based on the unit interval $[0, 1]$ such that $y - mx = c$.*

Proof. Consider the Cantor set on the unit interval of the x-axis and on the unit interval of the y-axis of a rectangular cartesian coordinate system. Let M be the product of these two sets. That is, M consists of all points of the unit square $(0, 0)$, $(0, 1)$, $(1, 1)$, $(1, 0)$ having coordinates (x, y), each of which is a coordinate of a point of the Cantor set on $[0, 1]$. To prove the theorem we will show that for an admissible c and m, the line $y - mx = c$ meets the set M in at least one point (x, y).

Let S be any closed square in the unit square of the x, y-plane and having sides parallel to the coordinate axes. Divide S into three equal vertical strips by inserting two vertical line segments in S. Delete from S the interior of the middle strip as well as its top and bottom open segments. Now, delete the interior of a similar horizontal "middle third" strip so that there remain four closed squares, each of area one ninth the area of the original square S and separated by a region in the shape of a cross. It is easily seen that if $y=mx+a$ meets S and if $\frac{1}{3} \leqq |m| \leqq 3$, then the line meets at least one of the four smaller squares.

If $0 \leqq c \leqq 1$, the line $y=mx+c$ meets the closed unit square and if $\frac{1}{3} \leqq |m| \leqq 3$, the line meets at least one of the four squares that remain after the first cross is deleted corresponding to the first step in the formation of the Cantor sets on the two axes. Fix the attention on one of the squares A, having a point in common with $y=mx+c$. After deletions corresponding to the second step in the formation of the Cantor sets on the two axes the line will still have a point in common with at least one of the smaller squares $B \subset A$. The proof is completed by induction and an application of the theorem that a monotone decreasing sequence of non-empty compact sets is non-empty.

This theorem yields Randolph's Theorems 1 and 2 (of the first paper cited) in case $|m|=1$.

A THEOREM CONCERNING FUNCTIONS DISCONTINUOUS ON A DENSE SET

M. K. Fort, Jr., University of Illinois

1. Introduction. It is well known that a function can be discontinuous at points of an everywhere dense set, and yet be continuous at points of a residual set. One might wonder whether or not it is possible for such functions to be differentiable at points of a residual set. The theorem proved in this note shows that this is not possible.

In the problem section of a recent issue of this Monthly, C. D. Olds proposed the problem of finding an example of a function which is discontinuous in an everywhere dense set and which is also differentiable in an everywhere dense set.* It follows from our theorem that functions of this type have a somewhat surprising property; namely, at "most" points (*i.e.* on a residual set) the function is continuous but not differentiable.

We let f be a real valued function which is defined for all real numbers. It will be convenient to let A be the set of points at which f is discontinuous, B be the set of points at which f is differentiable and C be the set of points at which f is continuous but not differentiable.

2. Theorem. We shall prove the following

Theorem. *If A is everywhere dense, then B is a set of the first category.*

* Vol. 57 (1950), p. 557.

Proof. For each positive integer n, we define D_n to be the set of all points p which have the property that there exist points x and y in the $1/n$-neighborhood of p for which

$$\frac{f(x) - f(p)}{x - p} - \frac{f(y) - f(p)}{y - p} > 1.$$

Next we define $D = \bigcap_{n=1}^{\infty} D_n$.

It is clear that f is not differentiable at any point of D. If we can show that D is a residual set, it follows that B must be a set of the first category. Hence, in order to establish our theorem it is sufficient to prove that D is a residual set.

In order to prove that D is a residual set, it is sufficient to prove that for each n the set of interior points of D_n form an everywhere dense set. Let I be an open interval of real numbers. Since A is everywhere dense, there exists $q \in I \cap A$. We now choose real numbers h and k such that

$$\max (f(q), \limsup_{t \to q} f(t)) > h > k > \min (f(q), \liminf_{t \to q} f(t)).$$

Now suppose $p \in I$, $q - 1/n < p < q$ and $h - k > q - p$. We obtain

$$\frac{h - f(p)}{q - p} - \frac{k - f(p)}{q - p} = \frac{h - k}{q - p} > 1.$$

It is possible to find x and y arbitrarily close to q for which $f(x) > h$ and $f(y) < k$. Hence it is possible to find x and y in the $1/n$-neighborhood of p for which

$$\frac{f(x) - f(p)}{x - p} - \frac{f(y) - f(p)}{y - p} > 1.$$

This proves that $p \in D_n$. Since the set of all $p \in I$ for which $q - 1/n < p < q$ and $h - k > q - p$ forms an open set, each such p is interior to D_n. It follows that the interior of D_n is everywhere dense.

The following corollary is of interest in connection with functions of the type suggested by Olds in his problem.

COROLLARY. *If A and B are both everywhere dense, then C is a residual set (and hence also everywhere dense).*

Proof. Since f is differentiable at points of an everywhere dense set, f is certainly continuous at points of an everywhere dense set. It follows from a well known theorem that A must be a set of the first category. The above theorem proves that B is a set of the first category. Since C is the complement of $A \cup B$, C must be a residual set.

3. An example. Let $x_1, x_2, x_3, \cdots$ be an enumeration of the set R of rational numbers (or any other countable everywhere dense set). We define $f(x_n) = 1/n^3$ for each positive integer n, and define $f(t) = 0$ if $t \notin R$. It is obvious that f is discontinuous at each point of R, and continuous at every other point.

We let S_n be the open interval $(x_n-1/n^2, x_n+1/n^2)$. It is easy to see that the set

$$S = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} S_k$$

has measure zero, and hence that $S \cup R$ has measure zero. If $q \notin S \cup R$ then $p \notin (x_n - 1/n^2, x_n+1/n^2)$ for sufficiently large n, and hence

$$\left| \frac{f(x_n) - f(p)}{x_n - p} \right| \leqq \frac{1}{n}$$

for all sufficiently large n. It follows that f is differentiable at each point of the complement of $S \cup R$. Since $S \cup R$ has measure zero, f is differentiable not only at points of an everywhere dense set but almost everywhere.

It is interesting to note that although our theorem proves that C has "more" points than B in the sense of category, this example shows that B can be "larger" than C in the sense of measure theory.

CLASSROOM NOTES

EDITED BY C. B. ALLENDOERFER, Haverford College

All material for this department should be sent to G. B. Thomas, Department of Mathematics, Massachusetts Institute of Technology, Cambridge 39, Mass.

FURTHER NOTE ON THE REMAINDER IN COMPUTING BY SERIES

R. K. MORLEY, Worcester Polytechnic Institute

This note is a supplement to the author's paper on this topic in the October, 1950, number of the MONTHLY.

As there we denote a series of positive terms by $\sum_{n=1}^{\infty} u_n$, and we suppose that $u_n = f(n)$ is such that $f(n)$ can be changed to $f(x)$ with x a continuous variable. We suppose that the sum of m terms of the series has been computed and we assume that $\int_m^{\infty} f(x)dx$ is finite and can be found, and that for $m \leqq x$, $f'(x)$ is negative and $f''(x)$ is positive. We let $R = \sum_{n=m+1}^{\infty} u_n$.

In the previous paper lower and upper bounds for R were found as follows:

$$R > \int_m^{\infty} f(x)dx - \frac{1}{2} u_m \tag{1}$$

$$R < \int_m^{\infty} f(x)dx - \frac{1}{2} u_{m+1}. \tag{2}$$

The purpose of the present note is to obtain somewhat closer bounds.

As before we draw the curve $y=f(x)$ and represent the terms u_n by rectangles of base 1 under the curve whose right-hand sides are $f(n)$, and also by rectangles of base 1 above the curve whose left-hand sides are $f(n)$. We denote the points $[m+1, f(m+1)]$, $[m+2, f(m+2)]$, $[m, f(m+1)]$, $[m+2, f(m+1)]$ by B, C, D, G respectively. Draw chord BC. Also draw a tangent to the curve at B and produce it till it cuts $x=m$ at H.

For the lower bound for R we consider $u_{m+1}=$ the rectangle under BG.

$$u_{m+1} > \int_{m+1}^{m+2} f(x)dx + \triangle BCG, \qquad \triangle BCG = \frac{1}{2}(u_{m+1} - u_{m+2})$$

and we have similar expressions for the other terms of R.

Hence

$$R > \int_{m+1}^{\infty} f(x)dx + \frac{1}{2}[(u_{m+1} - u_{m+2}) + (u_{m+2} - u_{m+3}) \cdots]$$

or

$$R > \int_{m+1}^{\infty} f(x)dx + \frac{1}{2}u_{m+1}. \tag{3}$$

It is evident that this is the same as adding u_{m+1} to the lower bound for $\sum_{m+2}^{\infty} u_n$ given by (1). (3) gives a slightly closer lower bound for R than (1).

For the upper bound for R we consider $u_{m+1}=$ the rectangle under DB. With the restrictions assumed for $f(x)$ the tangent BH is wholly below the curve. Therefore

$$u_{m+1} < \int_{m}^{m+1} f(x)dx - \triangle DBH \tag{4}$$

and we have similar expressions for the other terms of R.

$$\triangle DBH = \frac{1}{2}[-f'(m+1)].$$

Denote triangles like DBH under tangents at $m+1$, $m+2$, etc. by T_{m+1} etc. Then $T_i = -\frac{1}{2}f'(i)$. By (3) we have

$$\sum_{i=m+1}^{\infty} T_i > \frac{1}{2}\left\{\int_{m+1}^{\infty}[-f'(x)]dx + \frac{1}{2}[-f'(m+1)]\right\}$$

or

$$\sum_{i=m+1}^{\infty} T_i > \frac{1}{2} f(m+1) + \frac{1}{4}[-f'(m+1)]$$

$$> \frac{1}{2} u_{m+1} + \frac{1}{4}[-f'(m+1)]. \tag{5}$$

From (4)

$$R < \int_m^{\infty} f(x)dx - \sum_{i=m+1}^{\infty} T_i$$

and applying (5) to this we have for the wanted lower bound for R

$$R < \int_m^{\infty} f(x)dx - \frac{1}{2} u_{m+1} - \frac{1}{4}[-f'(m+1)]$$

It will be noted that this is the same as the bound that was given by (2) less $\frac{1}{4}[-f'(m+1)]$. It therefore gives a closer bound. In some cases the improvement is considerable.

ALGEBRAIC NUMBERS AND A POINT OF VIEW

R. M. REDHEFFER, University of California at Los Angeles

Introduction. The rational numbers are closed under the arithmetical operations; one cannot get a new number by addition, subtraction, multiplication, or division. On the other hand, if one adjoins to these arithmetical operations the "analytical" operation of taking a limit, then the rationals are no longer closed. There arises a larger set (the real numbers) which includes $\sqrt{2}$, among others.

Consider that new set under the five operations, addition, subtraction, multiplication, division, and taking limits. Is it closed? Offhand, one might be inclined to think not. To be sure, the limit of every *rational* sequence will be real, since the reals were obtained that way. But why should the limit of every real sequence be real? As in the theory of Baire classes, one might expect an infinite hierarchy of more and more complicated numbers.

In spite of one's doubts this does not happen; the reals are closed under the five operations, and we must turn elsewhere to obtain anything new. *Such is the meaning of the Dedekind theorem.* As soon as the rationals are thought of as points on a line, rather than all jumbled up together in a set with no obvious geometrical structure—as soon as this is done, our failure to get anything new becomes perfectly understandable.

Inverse. The notion of adjoining an operation gives an interesting way of looking at algebraic numbers. Instead of "limit," this time, we adjoin the operation of "inverse." If the polynomial $P(z)$ belongs to a field K, in the sense that all its coefficients are in K, then $w=P(z)$ will be in K whenever z is. Thus, given z we can find w. On the other hand, for w in our set there may or may not be a z such that $P(z)=w$; given w, we can *sometimes* find z. On other occasions

we shall get something new. If by chance at least one root of $P(z)=w$ is in K, for all $w\in K$ and all $P\in K$, then we say that the field K has *inversive closure*. Similarly when K is only a set, not a field. Instead of polynomials $P(z)$ we shall sometimes consider the most general function that can be constructed with the operations available.

Since we speak here of complex numbers, this notion of inversive closure surely makes sense—it is known what we mean by the roots of $P(z)=w$, whether or not they are in K. On the other hand, to hunt for the inverse of a given element is a new operation, different from the field operations that we had to start with.

Rational numbers. On the most naive level it is exactly thus that the number system developed. With *counting* as fundamental, one readily defines "+." It is possible then to form a function as complicated as $a+z$, both a and z being integers. If $w=a+z$, then w is in the set whenever z is. On the other hand given w we may or may not be able to find z; the new things that arise thus are the negative integers. Similarly, the solution of $az-b=0$ gives the fractions, while the solution of $z^2+1=0$ gives the imaginaries.

Addition allows one to form not only $a+z$ but also $a+z+z+\cdots+z=a+bz$. If now we hunt for a solution z to the equation $a+bz=w$, we get all the positive and negative rationals. Hence the set becomes a field, and there is inversive closure, in that the set contains all inverses of the most complicated functions that can be formed.

Linear equations. These remarks presuppose that the integers 1, 2, 3, $\cdots$ are known to be in the set. What can be said when inversive closure alone is postulated? We confine ourselves at first to the linear function $a+bz$, and make the following observations:

I. Suppose given a set of complex numbers, at least one of which is different from zero. If the root z of $a+bz=0$ is in the set whenever a and b are, then the set need not form a field. But if $z+1$ is in the set whenever z is, then the set must be a field. And hence if the root of $a+bz=w$ is in the set whenever a, b, w are, then the set is a field.

To see this, note first that the set consisting of -1 alone satisfies the first closure condition, as does also the set consisting of -1 and 1. For the second part of the assertion, the choice $a=b\neq 0$ shows that -1 is in the set, whence by the $z+1$ condition we know that 0 and 1 are. It is possible then to form $a+z=0$, so that $-a$ is in the set whenever a is. Since negatives are in the set we can consider $az-b=0$ to see that b/a is in the set whenever b and $a\neq 0$ are. In particular $1/b$ is in the set, and hence the root ab of $(1/b)z-a=0$ is in the set. Finally, the presence of b/a insures that of $b/a+1$, by our special condition $z+1$; and product closure shows then that $a(b/a+1)=b+a$ is in the set too.

Turning now to the third part of the statement, we note that $az+a=a$ makes the set contain 0. Then $az+0=a$ shows that it contains 1, while az

$+a=0$ shows that it contains -1. We can now form $z-1=a$, to get the $z+1$ condition needed to complete the proof.

Since the solution of $a+bz=w$ is rational whenever a, b and w are, it is evident that the rational field has inversive closure in our present sense of the word (addition alone being permissible when forming functions of z). The result just established says that no smaller set can be closed in this sense.

Polynomial equations. The function $a+z$ involves a single addition and gives us the negatives. The function $a+bz$ involves repeated addition for integer b, or addition and one multiplication in general. It gives us the positive and negative rationals, and this whether or not the presence of the integers is postulated beforehand. Similarly one might consider az, formed by a single multiplication.

Leaving these special functions, let us proceed at once to full use of both addition and multiplication, with repeats. One obtains the set of all polynomials $P(z)$ with coefficients in the set, and one is led to $P(z)=w$ as the most general equation requiring an inverse. It is evident that the rationals do not have inversive closure in this extended sense, and yet the set of all complex numbers has it. Such is the meaning, for us, of the Gauss theorem on polynomial equations.

Since the rationals are not closed while the set of all complex numbers is closed, one might expect that a process of adjoining suitably chosen numbers to the rationals should eventually lead to a minimal closed set. It will be found that this is the case. There is a set of numbers which is a proper subset of all the complex numbers, which is closed under polynomial inversion, and which is minimal in that every closed set must contain this one as a subset.

Algebraic numbers. Let us see, now, how this minimal closed set (under polynomial inversion) is to be constructed. By I, the special case in which $P(z)$ is linear gives us the positive and negative rationals, at least. Hence we can form all polynomials with rational coefficients, and in particular, we can form the equations $P(z)=0$ where $P(z)$ has integer coefficients. The "inverses" z under this transformation are called *algebraic numbers*. It is clear that they are a notable enrichment of the number system; an equation as simple as $z^2-2=0$ already implies the idea inherent in *limit*, while $z^2+1=0$ gives something strange indeed.

Though we postulate only that $P(z)=0$ shall have one root in the set, it is easy to see that the equation must in fact have all roots in the set. For the set is known, by I, to be closed under the field operations. Hence the coefficients of $Q(z)=P(z)/(z-r)$ will be in the set whenever r is and $P(r)=0$. Therefore $Q(z)$ has a root in the set, and the result follows by repetition. Thus it is that any set closed under polynomial inversion must contain all the algebraic numbers, at least (provided always it is not the set 0 alone). If the algebraic numbers are themselves closed, then our task is finished.

By repeated use of Sylvester's dialytic procedure* we can show rather simply that an equation $P(z)=0$ with algebraic numbers for coefficients has roots which are again algebraic numbers. Moreover if z is algebraic, so is $z+1$; in fact $z+1$ is a root of $Q(z)=P(z-1)=0$ whenever $P(z)=0$, and $Q(z)$ obviously has integer coefficients. Therefore by I we see at once that the algebraic numbers form a field. In particular the sum of two algebraic numbers is algebraic, and hence the equation $P(z)=w$, with algebraic coefficients in P, reduces to $Q(z)=0$ with algebraic coefficients in Q. The result just quoted shows that all the roots are algebraic, and closure is thus established. All this can be summarized as follows:

II. Given a set of complex numbers at least one of which is different from zero. Suppose a root of $az^n+bz^{n-1}+\cdots+k=w$ *is in the set whenever* $a, b, \cdots, k, w$ *are. Then the set contains not only one but all roots of these equations. Also it contains the rationals, hence the algebraic numbers. These latter have the closure property, so that the algebraic numbers could be defined as the unique minimal set closed under polynomial inversion. The algebraic numbers form a field, hence are closed under the arithmetic operations as well.*

Dedekind analogue. Starting with the positive integers under addition and multiplication one forms the polynomials in z, that is, the most general functions that addition and multiplication allow. To the operations of addition and multiplication one now adjoins the further operation of polynomial inversion. In this way one obtains the algebraic numbers. If new polynomial equations are formed with algebraic numbers for coefficients, it might be expected that the roots would lead to still other numbers. The result just stated says that this does not happen, but that the set of algebraic numbers is closed under the operations of addition, multiplication, subtraction, division, *and inverse.* Here we have the algebraic analogue of Dedekind's theorem as discussed previously. The closure and minimal property which Dedekind theory establishes for the continuum corresponds to the closure and minimal property established now for the algebraic numbers.

Inverse images of zero. The algebraic numbers appear as solutions of the equations $P(z)=0$, and do not require the more general equations $P(z)=w$. By I it is known that these two kinds of inverse are not equivalent when $P(z)$ is linear, and we shall see how they are related in the general case.

III. Suppose given a set of complex numbers closed in the sense that a root of $az^n+bz^{n-1}+\cdots+k=0$ *is in the set whenever* $a, b, \cdots, k$ *are. Then the set need not be a field. But if* $z+1$ *is in the set whenever* z *is, then the set is a field. If the set is a field then it will be closed in the sense* $az^n+bz^{n-1}+\cdots+k=w$, *and conversely closure in this sense always implies that the set is a field. If the set is a field with a non-zero element then it must contain the algebraic numbers as subfield.*

* Philip Franklin, Treatise on Advanced Calculus, John Wiley and Sons, 1940, Problem 30. This problem sequence suggested the present discussion.

To see this, note that the roots of $z^n+bz^{n-1}+\cdots+k=0$ will all be algebraic numbers, integers or units according as the coefficients $b, \cdots, k$ are all algebraic numbers, integers, or units. Again, the quotient of two units is a unit. Hence the set of algebraic units is closed in the sense that $P(z)=0$ has all its roots in the set, and yet it is not a field. The other assertions in *III* follow from *I*.

The generality of the function. Speaking of fields we have available the operation of division. Hence the meaningful functions are the rational functions $R(z)$, not just the polynomials, and the most general equation that makes sense, with the operations at our disposal, is $R(z)=w$. Now, even the set of all complex numbers need not contain solutions to an equation of this form, as is shown by the example $1/z=0$. Since inversive closure in the previous sense is out of the question, we ask instead that $R(z)=w$ have a solution *for all but one* w. The exceptional value happens to be identifiable as $w=0$; in other cases the equation $R(z)=w$ reduces to a polynomial equation in z. Thus closure still holds, both for the complex numbers and for the algebraic numbers, an exceptional w being admitted.

Closure with these "most general constructible" functions $R(z)$ is the strongest type of closure that one could reasonably ask of a field, and the algebraic numbers have it. We use the field operations in forming functions $R(z)$; we ask for a minimal field such that $R(z)=w$ always has a solution, except perhaps for one w; and we find the algebraic numbers as unique set with these properties.

By using inversion one can construct still more general functions (the algebraic functions, among others). Also for complex fields there is defined another arithmetic operation, raising to a power. Thus 2^z has a meaning. Using both exponentiation and inversion one is led to a large class of functions, for which the algebraic numbers fall far short of inversive closure. The Gelfond theorem concerning transcendence of α^β shows that an equation as simple as $2^z=w$ fails unless w is a rational power of 2.

Whether or not all the complex numbers are closed (or in some sense nearly closed) under these functions seems difficult to determine. It is true that the singularities are isolated, as can be shown by induction; and hence for all single-valued functions and for all but one w the Picard theorem guarantees existence of a z such that $F(z)=w$. But the elliptic modular function and other examples show that the assumption of single-valuedness is essential, if we wish to avoid detailed study.

One can bypass the question by seeking a minimal set that contains a root of $F(z)=w$ whenever the set of all complex numbers does so. Here $F(z)$ is any function formed by use of a finite number of additions, subtractions, multiplications, divisions, exponentiations, and inversions upon the indeterminate z and the numbers in the minimal set.

The set of roots of $F(z)=w$ is countable. Also if the set of numbers at hand is countable, then the set of functions F is likewise countable; the proof resembles

that used to show that there are only countably many algebraic numbers. Start with the set E_1 of integers, and add to this the roots z of all $F(z)=w$ formed by the above operations on members of E_1 and on z. There arises a new countable set, E_2. Now form all equations $F(z)=w$ by use of members of E_2 and the basic operations, and adjoin their roots to E_2. This gives a countable set E_3, and so on. The minimal set such that it contains every root of $F(z)=w$ is $E_1+E_2+E_3+\cdots$, whilst the minimum set such that it contains at least one root of $F(z)=w$ is $E_1+E_2+E_3+\cdots$ or a subset of it. In either case the set is countable.

What does all this show? It shows that the operation of inversion is essentially different from, and in some sense weaker than, the operation of limit. Even when carried to extremes, *inversion* gives an enumerable set at most, while *limit* gives the continuum.

ELEMENTARY PROBLEMS AND SOLUTIONS

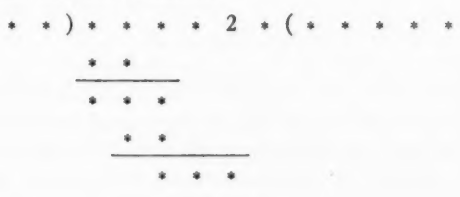
EDITED BY HOWARD EVES, Oregon State College

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Oregon State College, Corvallis, Oregon. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 971. *Proposed by A. W. Willcox, Washington, D. C.*

Reconstruct the division problem:



E 972. *Proposed by Michael Goldberg, Washington, D. C.*

Dissect a regular pentagon, by straight cuts, into six pieces which can be put together to form an equilateral triangle. (See Ball-Coxeter, *Mathematical Recreations*, top of p. 93.)

E 973. *Proposed by Leo Moser, Texas Technological College*

Show that if

$$x = -(b/a)\cos^2\{(1/4)\cos^{-1}(\{b^2 - 8ac\}/b^2)\}$$

exists, then it is a solution of $ax^2+bx+c=0$.

E 974. *Proposed by Alan Wayne, Flushing, N. Y*

Let L, M, N divide the sides BC, CA, AB of triangle ABC in the ratios $BL/LC=r$, $CM/MA=s$, $AN/NB=t$. Let R, S, T be the intersections of the pairs of cevians BM, CN; CN, AL; AL, BM. Show that the cross-ratios (AL, ST), (BM, TR), (CN, RS) are all equal to rst.

E 975. *Proposed by M. R. Spiegel, Rennselaer Polytechnic Institute*

Prove that

$$\int_0^1 dx/x^x = \sum_{i=1}^{\infty} 1/i^i.$$

SOLUTIONS

Concurrent Spheres

E 940 [1950, 632]. *Proposed by Victor Thébault, Tennie, Sarthe, France.*

Being given a tetrahedron $ABCD$, an arbitrary point P, and four coplanar points A', B', C', D' lying in the planes of the faces BCD, CDA, DAB, ABC, show that the lines PA', PB', PC', PD' intersect the spheres $PBCD$, $PCDA$, $PDAB$, $PABC$ in four points cospherical with P.

Solution by Howard Eves, Oregon State College. Let PA', PB', PC', PD' intersect the spheres $PBCD$, $PCDA$, $PDAB$, $PABC$ in the points A'', B'', C'', D''. Let (S) denote the sphere through A, B, C, D and (S'') the sphere through P, A'', B'', C''. Since A' lies on the radical plane of (S) and $(PBCD)$ and also on the radical plane of (S'') and $(PBCD)$, it lies on the radical plane of (S) and (S''). Similarly, B' and C' lie on the radical plane of (S) and (S''). It follows that D' is on the radical plane of (S) and (S''). But D' is on the radical plane of (S) and $(PABC)$. It is therefore on the radical plane of (S'') and $(PABC)$, and D'' is on (S''). Thus P, A'', B'', C'', D'' are cospherical.

The analogous theorem in the plane may be similarly established.

Also solved by Joseph Langr and Roger Lessard.

Periodic Properties of the Digits in the Powers of 2

E 942 [1950, 686]. *Proposed by L. A. Ringenberg, Eastern Illincis State College*

In the sequence of powers of 2:2, 4, 8, 16, 32, $\cdots$, the units digits form a sequence of period four. What periodic properties do the other digits have?

Solution by Aaron Buchman, Buffalo, New York. Let the units digit be called the *first* digit; the tens digit, the *second* digit; *etc.* Then the period, k, of the periodic part of the sequence formed by the nth digits will be the smallest value of k which satisfies the congruence

$$2^{s+k} \equiv 2^s \pmod{10^n},$$

or

$$2^{s+k-n} \equiv 2^{s-n} \pmod{5^n}.$$

Putting s equal to its minimum value of n, we have

$$2^k \equiv 1 \pmod{5^n}.$$

Hence, by Euler's theorem,

$$k = \phi(5^n) = 4(5^{n-1}).$$

It is evident that the non-periodic pre-period of the sequence formed by the nth digits consists of $n-1$ zeros.

By a similar proof we may show that the period, k, of the nth digits of the powers of an integer a expressed in the number system with base $b>a$ is

$$k = \phi(\{b/g\}^n),$$

where $g=(a, b)$. Again, the periodic part of the sequence is preceded by $n-1$ non-periodic digits.

Also solved by J. H. Braun, D. H. Browne, R. E. Gettig, Roger Lessard, Theodore Lindquist, Fred Marer, Prasert Na Nagara, C. S. Ogilvy, and the proposer.

Central Motion

E 943 [1950, 686]. *Proposed by S. H. Gould, Purdue University.*

By analogy with the motion of the planets, it seems natural to assume for any central motion that, at least if the particle never passes through the center, (a) the distance of the particle from the center is a maximum only when the velocity of the particle is a minimum, (b) the velocity is a maximum only when the distance is a minimum. Prove (a) and give a counter-example of (b).

Solution by H. D. Block and C. E. Langenhop, Iowa State College. We shall not only prove what is asked but shall obtain the conditions under which the "natural assumptions" are valid and those under which the anomalous behavior described in (b) can occur. We do not assume that the force is derivable from a potential or is constant in time.

Using polar coordinates let $f(r, \theta, t)$ be the force toward the center so that $f>0$ is an attraction and $f<0$ a repulsion. Then, using the radial and trans-

verse components of acceleration and Newton's law, we have:

$$-f(r, \theta, t) = m(\ddot{r} - r\dot{\theta}^2), \tag{1}$$

$$m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = (m/r)d(r^2\dot{\theta})/dt = 0. \tag{2}$$

From (2) it follows that $r^2\dot{\theta} = h$, where h is constant during the motion. Then (1) becomes

$$-f(r, \theta, t) = m(\ddot{r} - h^2/r^3). \tag{3}$$

Now $v^2 = \dot{r}^2 + r^2\dot{\theta}^2 = \dot{r}^2 + h^2/r^2$, so that

$$d(v^2)/dt = 2\dot{r}\ddot{r} - 2h^2\dot{r}/r^3 = 2\dot{r}(\ddot{r} - h^2/r^3). \tag{4}$$

Comparing (3) and (4) we see that we have

$$d(v^2)/dt = -(2\dot{r}/m)f(r, \theta, t) = -(2/m)f(r, \theta, t)(dr/dt), \tag{5}$$

which might also be obtained directly from the theorem that the change in kinetic energy equals the work done.

We assume that the time parameter in the orbit can always be extended in both directions, and that the motion is sufficiently smooth so that the conditions of maxima and minima are given by a change in sign of the time derivative. Inspection of (5) then leads immediately to

THEOREM I (a) *If $f(r_1, \theta_1, t_1) > 0$, then at (r_1, θ_1, t_1)* (i) *v is a minimum if and only if r is a maximum*, (ii) *v is a maximum if and only if r is a minimum*. (b) *If $f(r_1, \theta_1, t_1) < 0$, then at (r_1, θ_1, t_1)* (iii) *v is a minimum if and only if r is a minimum*, (iv) *v is a maximum if and only if r is a maximum*.

The condition (iv), however, cannot occur, as one sees from

THEOREM II. *If $f(r_1, \theta_1, t_1) \leqq 0$, then r is not a maximum at (r_1, θ_1, t_1).*

This is easily seen by rewriting (3) as $\ddot{r} = h^2/r^2 - f(r, \theta, t)/m$, so that $\ddot{r} > 0$ in this case. Then, if $\dot{r} = 0$, r must be a minimum.

To establish the results asked for in the problem we note that if r is a maximum then, by Theorem II, $f > 0$ and, by Theorem I (a), v is a minimum. As for a counterexample for the assertion (b) we can take the zero force field. Here the particle travels in a straight line with constant velocity, and with respect to any center not lying on the line of motion v is always a maximum or a minimum while r, in general, is neither. A second counterexample which is less trivial is that of a mass on a spring given an initial velocity in the direction of the spring, with the center of attraction considered to be a point on the extension of the line of the spring. Here the maximum v occurs at an intermediate value of r.

In the two counterexamples considered it may be observed that $f = 0$ at the point where the anomalous behavior occurs. That this must be the case follows from our results. For if v is a maximum (r_1, θ_1, t_1), then $f(r_1, \theta_1, t_1) \geqq 0$. But if $f(r_1, \theta_1, t_1) > 0$ then, by Theorem I (a)–(ii), r is a minimum. Thus the

condition that v be a maximum and r not a minimum at (r_1, θ_1, t_1) can occur only if $f(r_1, \theta_1, t_1) = 0$.

Also solved by M. S. Klamkin, W. W. Zaring, and the proposer.

A Maximum Truncated Cone

E 944 [1950, 686]. *Proposed by E. R. Bowersox, Chicago, Ill.*

A truncated cone with one end open is to be formed from a circular sheet, cutting out a central portion for the base, and using the remaining annular ring for the curved area. Solve for the truncated cone of maximum volume.

Solution by Frank Herlihy, Comstock, Mich. Assume the given sheet has unit radius and let r denote the radius of the piece to be cut out. Then the dimensions of the truncated cone are easily found to be: radius of lower base $= r$, radius of upper base $= r^2$, slant height $= 1 - r$, altitude $= (1 - r)(1 - r^2)^{1/2}$. We then find the volume to be

$$V = (\pi r^2/3)(1 - r^3)(1 - r^2)^{1/2}.$$

Equating dV/dr to zero we get

$$r(r - 1)(6r^4 + 6r^3 + r^2 - 2r - 2) = 0.$$

The roots $r = 0$ and $r = 1$ lead to minimum volumes. The quartic factor has only one positive root, which is the value of r leading to the maximum volume. We find that $r = 0.66148+$, yielding $V_{\max} = 0.2242-$.

Also solved by J. H. Braun, D. H. Browne, David Mandelbaum, and C. S. Ogilvy.

Trihedral Angles of a Polyhedron

E 945 [1950, 687]. *Proposed by Leo Moser, University of North Carolina*

If all the faces of a convex polyhedron have central symmetry show that there are at least eight vertices where exactly three edges meet. (The cube has exactly eight such vertices.)

Solution by E. A. Nordhaus, Michigan State College. Consider any polyhedron for which the Euler-Descartes formula $F + V = E + 2$ is valid, where F, V, and E are respectively the number of faces, vertices, and edges. Convexity and central symmetry will not be employed, and the result stated will be shown true under the weaker assumption that no face is triangular. Let $F_i (i \geqq 3)$ denote the number of faces having i sides, and $V_j (j \geqq 3)$ the number of vertices where exactly j edges meet, so $F = \Sigma F_i$, $V = \Sigma V_j$. Since each edge is common to two faces and terminates in two vertices, $2E = \Sigma i F_i = \Sigma j V_j$. If we multiply the Euler-Descartes relation by 4 and insert the above values, then

$$V_3 + F_3 = 8 + (F_5 + V_5) + 2(F_6 + V_6) + \cdots .$$

The desired result follows when $F_3 = 0$. We also notice a duality between faces and vertices, since $F_3 \geqq 8$ if $V_3 = 0$.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten, with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4434. [1951, 266]. *Correction*

Replace the upper limit 1 in the integral by π. Note also that r and n are positive integers.

4443. *Proposed by R. H. Breusch, Amherst College, Massachusetts*

If

$$P_n(z) \equiv \sum_{p=0}^{[n/2]} (-1)^p \frac{z^{n-2p}}{(n-2p)!} = \frac{z^n}{n!} - \frac{z^{n-2}}{(n-2)!} + \cdots,$$

prove that the number of real zeros of $P_n(z)$ is

$$N_n = \frac{2n}{e\pi} + \frac{\log n}{e\pi} + O(1).$$

4444. *Proposed by J. H. Braun, Illinois Institute of Technology*

Prove that no three of the diagonals of a regular polygon of odd order are concurrent at any point other than the vertices.

4445. *Proposed by Paul Erdös, University of Aberdeen, Scotland*

Split the set of primes $p_1 < p_2 < \cdots$ into two classes q_i and r_i so that $\sum 1/q_i = \sum 1/r_i = \infty$. Define $\mu'(k) = 0$ if k is a multiple of one of the r's, otherwise $\mu'(k) = \mu(k)$, where $\mu(k)$ is the Moebius symbol (0 if k has a square factor, $+1$ if k has an even number of distinct prime factors, -1 if it has an odd number). Prove that

$$\sum_{k=1}^{\infty} \frac{\mu'(k)}{k} = 0.$$

4446. *Proposed by Victor Thébault, Tennie, Sarthe, France*

If the lines which join the vertices of a tetrahedron to the circumcenter (or the orthocenter) of the opposite faces form a hyperbolic group, the tetrahedron is equifacial and conversely. If the lines are concurrent, the tetrahedron is regular.

4447. *Proposed by D. J. Newman, New York University*

Let $f(z)$ be analytic in $|z| \leqq 1$. Suppose $|f(z)| \leqq 2$ there and $f(0) = 1$. Show that $f(z)$ has no zeros for $|z| \leqq \frac{1}{2}$.

SOLUTIONS

Determinant Evaluation

4367 [1949, 637]. *Proposed by F. E. Wood, University of Oregon.*

Evaluate the determinant

$$\begin{vmatrix} {}_nP_n & {}_nC_1 & {}_nC_2 \cdots & {}_nC_n \\ {}_{n-1}P_{n-1} & {}_{n-1}C_0 & {}_{n-1}C_1 \cdots & {}_{n-1}C_{n-1} \\ {}_{n-2}P_{n-2} & 0 & {}_{n-2}C_0 \cdots & {}_{n-2}C_{n-2} \\ \cdots & \cdots & \cdots & \cdots \\ {}_0P_0 & 0 & \cdots 0 & {}_0C_0 \end{vmatrix}$$

of the $(n+1)$th order where ${}_nP_r$ and ${}_nC_r$ are the usual permutation and combination symbols.

I. *Solution by G. Y. Cherlin, Rutgers University.* Noting that ${}_rC_0 = 1$ for all r, and denoting the given $(n+1)$th order determinant by D_n we obtain by expanding with respect to the elements of the first row,

$$D_n = {}_nP_n - \sum_{i=1}^{n} {}_nC_i \cdot D_{n-i}$$

or

$${}_nP_n = \sum_{i=0}^{n} {}_nC_i \cdot D_{n-i}.$$

Using this it is readily proved by induction that

$$D_n = \sum_{s=0}^{n} (-1)^{n-s} {}_nP_s = n! \sum_{i=0}^{n} (-1)^i / i!.$$

II. *Solution by Koichi Yamamoto, Kanazawa University, Japan.* Subtract the $(r+1)$th row from the rth row $(r = 1, 2, \cdots, n)$ to obtain a determinant whose last column is filled with zeros except the last element which is 1. We may then delete its last column and last row to get

$$\begin{vmatrix} \Delta(n-1)! & {}_{n-1}C_1 & {}_{n-1}C_2 \cdots & {}_{n-1}C_{n-1} \\ \Delta(n-2)! & {}_{n-2}C_0 & {}_{n-2}C_1 \cdots & {}_{n-2}C_{n-2} \\ \cdots & \cdots & \cdots & \cdots \\ \Delta 0! & 0 & 0 \cdots & {}_0C_0 \end{vmatrix}$$

where $\Delta r! = (r+1)! - r!$. This has the same form as D_{n-1} except for the first column which is symbolically multiplied by the difference operator Δ. We repeat this process n times to arrive at the 1-rowed determinant

$$D_n = \Delta^n 0! = n! - {}_nC_1(n-1)! + {}_nC_2(n-2)! - \cdots + (-1)^n {}_nC_n 0!$$
$$= n!\left\{1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^n \frac{1}{n!}\right\}.$$

This is the "rencontre number" and is asymptotic to $n!/e$.

Also solved by J. A. Bullard, R. H. Cole, J. C. Eaves, N. J. Fine, G. P. Henderson, P. M. Hummel, N. D. Lane, Roger Lessard, W. V. Parker and A. C. Cohen, Jr., C. L. Perry, Azriel Rosenfeld, N. T. Seely, Jr., O. E. Stanaitis, H. E. Stelson, Ernst Trost, Louise A. Wolf, and the Proposer.

Editorial Note. Several solvers noted the recurrence relations

$$D_n = nD_{n-1} + (-1)^n, \qquad D_{n+1} = n\{D_n + D_{n-1}\}.$$

Noted also was the fact that the method of evaluation is independent of the first column. If the element in the first column and jth row is a_j, the determinant has the value

$$\sum_{i=0}^{n} (-1)^i {}_nC_i a_{i+1}.$$

Stelson referred to Weihrauch, *Zeitschrift f. Math. u. Phys.* XIX, 1874, pp. 354–360 and Muir, *Theory of Determinants*, v. 3, p. 465. The Proposer hit upon the determinant while working with a problem analogous to no. 4146 [1946, 107].

Miquel Point, Direct Circular Transformation

4370 [1950, 420]. *Proposed by H. F. Sandham, Trinity College, Ireland*

A', B', C' are points on the opposite sides of a triangle ABC. The circles through $B'C'A$, $C'A'B$, $A'B'C$ intersect in M, the Miquel point, whose isogonal conjugate is M'. Prove that M', M are corresponding points under the direct circular transformation set up by A, A'; B, B'; C, C'.

Solution by R. Deaux, Faculté Polytechnique, Mons, Belgium. The pedal triangles $A_1B_1C_1$, $A_1'B_1'C_1'$ of M, M' have a common circumcircle ω whose center is the midpoint of MM', and it is well known that M is the center of similitude of the triangles $A'B'C'$, $A_1B_1C_1$; hence, in the Gaussian plane,

$$(A'B'C'M) = (A_1B_1C_1M)$$

and if A_1'', B_1'', C_1'' are diametrically opposite to A_1, B_1, C_1 on ω

$$(A_1B_1C_1M) = (A_1''B_1''C_1''M').$$

Denoting by ∞ the point at infinity and by $\overline{X}$ the conjugate of a number X, the inversion with center M' and which leaves ω unchanged gives

$$(A_1'' B_1'' C_1'' M') = \overline{(A_1' B_1' C_1' \infty)}.$$

Using finally Schick's theorem* we have

$$\overline{(A_1' B_1' C_1' \infty)} = (ABCM'),$$

whence

$$(A'B'C'M) = (ABCM').$$

Also solved by Howard Eves and the Proposer.

Editorial Note. Eves shows also that if O is the circumcenter of triangle $A'B'C'$, then M, O are corresponding points in the same direct circular transformation.

Odd Primes and Certain Powers of Consecutive Integers

4373 [1949, 695]. *Proposed by Victor Thébault, Tennie, Sarthe, France.*

In every system of numeration having a base B which is the product of distinct prime factors, each to the first power, and which is not divisible by a given odd prime number m nor by any prime number of the form $2km+1$, the mth powers of integers which terminate in the digits 0 to $B-1$ all have distinct units digits.

Editorial Note. This problem and the Proposer's solution, together with a generalization, are given in *Mathesis*, v. LIX, 1950, pp. 10–12. No other solutions have been received.

Divergent Series

4374 [1949, 695]. *Proposed by Paul Erdös, University of Aberdeen, Scotland.*

Let $d_n>0$, $D_n=\sum_{k=1}^{n} d_k$, $D_n\to\infty$. The well known theorem of Abel-Dini states that $\sum_{n=1}^{\infty} d_n/D_n=\infty$. Now let $f(n)>0$ be any increasing function which satisfies $\sum_{n=1}^{\infty} d_n/f(n)=\infty$, then

$$\sum_{n=1}^{\infty} \frac{d_n}{\max\{D_n, f(n)\}} = \infty.$$

I. *Solution by N. J. Fine, University of Pennsylvania.* The result is trivial unless $D_n>f(n)$ infinitely often. Assuming the latter, for any M we can find an $N>M$ such that $D_N>\max\{f(N), 2D_M\}$. Then, since $\max\{D_n, f(n)\}$ increases,

$$\sum_{n=M+1}^{N} \frac{d_n}{\max\{D_n, f(n)\}} \geqq \frac{1}{\max\{D_N, f(N)\}} \sum_{n=M+1}^{N} d_n = \frac{1}{D_N}(D_N - D_M) > \frac{1}{2}.$$

Hence the series diverges.

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* See R. Deaux, *Introduction à la Géometrie des nombres complexes*, p. 154, or *Mathesis*, LVIII, p. 142.

II. *Solution by J. B. Kelly, University of Wisconsin.* Put $D(n)$ for D_n. Let $v_n = \max\{D(n), f(n)\}$. We may suppose that infinitely many of the v's are D's. Put $D(n_0)=1$, and from the infinite sequence of those D's which are v's we extract a subsequence $D(n_1), D(n_2), \cdots$, such that $\sum_{k=0}^{\infty} D(n_k)/D(n_{k+1}) < \infty$. Let s be the sum for this convergent series.

Now let n vary from n_k+1 to n_{k+1}. The sequences $D(n)$ and $f(n)$ are increasing. If $v_n = D(n)$, then $v_n < D(n_{k+1})$. If $v_n = f(n)$, then $v_n \leqq f(n_{k+1}) \leqq D(n_{k+1})$. It follows that

$$\sum_{n=n_k+1}^{n_{k+1}} \frac{d_n}{v_n} \geqq \sum_{n=n_k+1}^{n_{k+1}} \frac{d_n}{D(n_{k+1})} = \frac{D(n_{k+1}) - D(n_k)}{D(n_{k+1})} = 1 - \frac{D(n_k)}{D(n_{k+1})}.$$

Therefore

$$\sum_{n=1}^{n_{k+1}} \frac{d_n}{v_n} \geqq \sum_{j=0}^{k} \left(1 - \frac{D(n_j)}{D(n_{j+1})}\right) \geqq k - s.$$

Hence the theorem. If always $D(n) \geqq f(n)$, then the above becomes a proof of the Abel-Dini theorem itself.

Also solved by Joshua Barlaz, J. G. Millar, Norman Miller, and T. A. Newton.

Editorial Note. The Proposer calls attention to the fact that the above result is the best possible in the following sense: Assume that $g(n) \to \infty$ and $\sum_{n=1}^{\infty} d_n / D_n g(n) = \infty$. Then it can be shown that there exists an $f(n)$ with $f(n+1) \geqq f(n)$ for which

$$\sum_{n=1}^{\infty} \frac{d_n}{f(n)} = \infty, \qquad \sum_{n=1}^{\infty} \frac{d_n}{\max\{D_n g(n), f(n)\}} < \infty.$$

These results should be compared with problem 4278, [1949, 423].

Uniformly Bounded Sums

4375 [1950, 41]. *Proposed by N. J. Fine, University of Pennsylvania.*

Let $((x)) = x - [x] - \frac{1}{2}$. Prove that the sums

$$\sum_{n=1}^{m} ((2^n x + \tfrac{1}{2}))$$

are uniformly bounded.

Solution by T. M. Apostol, California Institute of Technology. From the relation

$$[2y] = [y] + [y + \tfrac{1}{2}]$$

we obtain

$$((2y)) = ((y)) + ((y + \tfrac{1}{2})).$$

Taking $y=2^n x$, we have

$$((2^n x + \tfrac{1}{2})) = ((2^{n+1}x)) - ((2^n x)),$$

so that the sum in question telescopes to

$$((2^{m+1}x)) - ((2x)).$$

Since $|((y))| \leqq \frac{1}{2}$, the uniform bound required is unity.

Also solved by C. E. Buell, J. B. Kelly, M. S. Klamkin, Robert Steinberg, H. S. Zuckerman, and the Proposer.

Editorial Note. The Proposer met the above problem in connection with a study of expansions in series of Walsh functions. It would be of great interest to obtain information about

$$S_m(\alpha, x) = \sum_{n=1}^{m} ((2^n x + \alpha)), \qquad 0 < \alpha < 1.$$

Note also the following theorem due to M. Kac: *Let* $E_m(\alpha, \beta)$ *be the set of numbers* x *in* $(0, 1)$ *for which*

$$\alpha < \frac{\sqrt{2}}{\sqrt{m}} \sum_{n=1}^{m} ((2^n x)) < \beta.$$

Then the measure of $E_m(\alpha, \beta)$ *tends to*

$$\frac{1}{\sqrt{\pi}} \int_{\alpha}^{\beta} e^{-y^2} dy.$$

Tetrahedron and Hyperbolic Group of Lines

4376 [1950, 41]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

For a tetrahedron $ABCD$ let (A) be the sphere with center A and radius equal to the altitude AA', and let the tangent planes to (A) through the edges BC, CD, DB intersect in the point A_1. Let B_1, C_1, D_1 be analogously defined. If $ABCD$ is orthocentric, show that AA_1, BB_1, CC_1, DD_1 are concurrent at the isogonal conjugate of the orthocenter, and that otherwise AA_1, BB_1, CC_1, DD_1 constitute a hyperbolic group of lines.

Solution by L. M. Kelly, Michigan State College. The tangent planes through the edges BC, CD, DB are in fact the reflections of the base plane BCD in the respective side faces through A. It follows that the isogonal conjugate point of A lies on the three planes through the three edges BC, CD, DB which are reflections of the side faces through A in the base plane. That is, the isogonal point of A is the reflection of the vertex A in the base plane. Thus the altitude through A and the line A_1A are isogonal rays relative to the vertex A. Similarly the lines B_1B, C_1C, D_1D are isogonal rays of their respective altitudes. It follows from established theorems that if the altitudes meet in a point so do their

isogonal rays and that since the altitudes in general constitute a hyperbolic group so do their isogonal rays.

Also solved by the Proposer.

Editorial Note. The corresponding theorem for the plane is also true and reads: For a triangle ABC let (A) be the circle with center A and radius equal to the altitude AA', and let the tangents to (A) through B and C intersect in the point A_1. Let B_1, C_1 be analogously defined. Then AA_1, BB_1, CC_1 are concurrent at the circumcenter of ABC.

Jensen's Inequality

4379 [1950, 42]. *Proposed by W. B. Fulks, University of Minnesota*

If

(1) $\phi(t)$ is defined in $I: m \leqq t \leqq M$,

(2) for any a, b in I, $a \neq b$

$$\phi\left(\frac{a+b}{2}\right) < \frac{\phi(a)+\phi(b)}{2},$$

(3) p_i are non-negative numbers, $i = 1, 2, \cdots, n$,

(4) $t_i \in I$, not all t_i equal,

it has been shown that

$$\phi\left(\frac{\sum_{i=1}^{n} p_i t_i}{\sum_{i=1}^{n} p_i}\right) \leqq \frac{\sum_{i=1}^{n} p_i \phi(t_i)}{\sum_{i=1}^{n} p_i}.$$

See Polya and Szegö, *Aufgaben und Ersätzen aus der Analysis*, vol. 1, p. 53, problem 74.

Show that the relation is true without the equality sign.

Solution by W. S. Gustin, Indiana University. Clearly we may take: $n \geqq 2$, all t_i distinct, all $p_i > 0$, and $\sum_i p_i = 1$; whence $\sum_{ij} p_i p_j = 1$. The $<$ relation may be deduced from the $\leqq$ relation as follows:

$$\begin{aligned}
\phi\Big(\sum_i p_i t_i\Big) &= \phi\Big(\sum_{ij} p_i p_j (\tfrac{1}{2}t_i + \tfrac{1}{2}t_j)\Big) \\
&\leqq \sum_{ij} p_i p_j \phi(\tfrac{1}{2}t_i + \tfrac{1}{2}t_j) \\
&< \sum_{ij} p_i p_j (\tfrac{1}{2}\phi(t_i) + \tfrac{1}{2}\phi(t_j)) \\
&= \sum_i p_i \phi(t_i).
\end{aligned}$$

We remark that the conclusion holds if and only if ϕ is continuous in addition to satisfying (1) and (2). For example: let f be a continuous solution of (2) and let g be a discontinuous solution of (2) wherein $=$ replaces $<$ (such solutions are easily constructed by use of a Hamel basis); then the function $\phi = f+g$ satisfies (2) but not the conclusion of the problem.

Also solved by Robert Steinberg and the Proposer.

Iterated Summation

4380 [1950, 119]. *Proposed by P. A. Pizá, San Juan, Puerto Rico*

For arbitrary positive integers n and k let

$$S_1(n, k) = 1^k + 2^k + \cdots + n^k,$$

and put

$$S_{p+1}(n, k) = S_p(1, k) + S_p(2, k) + \cdots + S_p(n, k),$$

with $p = 1, 2, \cdots$. Thus $S_p(n, k)$ is the pth iterated sum, the sum of the sum of $\cdots$ the sum of the first n perfect kth powers.

Show that $S_p(n, k)$ is a polynomial in n of degree $p+k$ and determine the form of the polynomial for $k = 3, 4, 5$. If possible, determine the form for general k. (The case $k = 2$ is the subject of problem 22, *Mathematics Magazine*, vol. XXII, no. 1, p. 51.)

Solution by Roger Lessard, École Polytechnique, Montreal. Let b_i and c_i be arbitrary constants to be specified later, and let

$$x^k = \sum_{i=0}^{\infty} a_i \binom{x + b_i}{c_i} \qquad k = 0, 1, 2, \cdots.$$

If it is possible to solve this system of equations to obtain the values of the a_i, we will have

$$S_1(n, k) = \sum_{x=1}^{n} \sum_{i=0}^{\infty} a_i \binom{x + b_i}{c_i} = \sum_{i=0}^{\infty} a_i \sum_{x=1}^{n} \binom{x + b_i}{c_i}$$

$$= \sum_{i=0}^{\infty} a_i \binom{n + b_i + 1}{c_i + 1},$$

. . . . . .

$$S_p(n, k) = \sum_{i=0}^{\infty} a_i \binom{n + b_i + p}{c_i + p}.$$

The simplest form is given with $b_i = 0$, $c_i = i$, whence

$$a_i = \sum_{j=0}^{i} (-1)^{i+j}\binom{i}{j} j^k = \Delta^i 0^k = i!G_k^i,$$

$$x^k = \sum_{i=0}^{k}\binom{x}{i}\Delta^i 0^k = \sum_{i=0}^{k} G_k^i (x)^{(i)}.$$

In the above, $a_i = 0$ for $i > k$. The coefficients $\Delta^i 0^k$ are the differences of zero, and the G_k^i are the Stirling's numbers of the second kind. (See problem no. 4276 [1949, 347] where the notation ${}_nK_j$ is used for $\Delta^j 0^n$.) We have then

$$S_p(n, k) = \sum_{i=0}^{k} a_i\binom{n+p}{p+i} = \sum_{i=0}^{k}\binom{n+p}{p+i}\sum_{j=0}^{i}(-1)^{i+j}\binom{i}{j}j^k.$$

For $k=3$; $a_1=1$, $a_2=6$, $a_3=6$. For $k=4$; $a_1=1$, $a_2=14$, $a_3=36$, $a_4=24$. For $k=5$; $a_1=1$, $a_2=30$, $a_3=150$, $a_4=240$, $a_5=120$. It is easy to see that the ith term is a polynomial of degree $p+i$ in n, so that consequently $S_p(n, k)$ is a polynomial in n of degree $p+k$.

Another form which has some advantages may be obtained by taking $b_i = i-1$, $c_i = k$, so that

$$a_i = \sum_{j=0}^{i}(-1)^{i+j}\binom{k+1}{i-j}j^k = a_{k+1-i}, \qquad a_i = 0 \text{ for } i > k,$$

$$x^k = \sum_{i=0}^{k} a_i\binom{x+i-1}{k}.$$

We have then

$$S_p(n, k) = \sum_{i=0}^{k}\binom{n+p+i-1}{k+p}\sum_{j=0}^{i}(-1)^{i+j}\binom{k+1}{i-j}j^k.$$

Many other forms are possible but do not seem to provide anything more useful. The use of "central notation," as usual, cuts the number of terms in the expansion in half, but requires different formulas for k even, odd.

Also solved by M. S. Klamkin, Kovina Milosevich, N. T. Seely, and the Proposer.

Editorial Note. It is not difficult to show that, if k is even

$$S_p(n, k) = \frac{(n+p)!(2n+p)}{(p+k)!(n-1)!}\phi(n^2+np),$$

with the same form when k is odd except that the factor $(2n+p)$ is suppressed. ϕ is a polynomial of degree $[(k-1)/2]$ with integral coefficients.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y. and not to any of the other editors or officers of the Association.

Fundamentals of the Calculus. By D. E. Richmond. McGraw-Hill Book Company, Inc., New York. 1950. 11+233 pages. $3.00.

It is a pleasant surprise to find a freshman textbook in which the author has dared to abandon the conventional in order to present an organized, unified, rigorous, and logical mathematical development suitable for the liberal arts student who elects only one year of college mathematics. By keeping algebraic requirements simple, and by carefully selecting and organizing topics, the author carries the student in a systematic manner from the integer to the integral. In the process he develops in detail the number system, inequalities, sequences, and limits. This development is beautiful and clear, involving many ideas infrequently met in elementary texts, for example, the notion of deleted neighborhood. In addition, analytic trigonometry is developed from the complex numbers, while logarithmic and exponential functions, and the analytic geometry of the conics are presented as applications of the derivative.

The readability and clarity of the writing should enable the student to understand the text with minimum assistance. The problems are carefully integrated with the text to promote the understanding of it. While many teachers might prefer that standard notations had been used throughout, the author avoids some on the ground that the notations make unnecessary difficulties for the student.

In no sense does the book pretend to be a standard calculus, since many topics, including the integral as the limit of a sum, are omitted. It might well be used, however, in any course where only an introduction to calculus is desired. The title may cause the book to be overlooked by teachers hunting a first year mathematics text. That would be unfortunate. Teachers who are weary of the many current texts, revised and enlarged, containing abundant material which is readily adaptable, but which may be omitted, should welcome this closely knit volume.

ROTHWELL STEPHENS

The Theory of Algebraic Numbers. By Harry Pollard. The Carus Mathematical Monographs, Number 9. Published by the Mathematical Association of America. Distributed by John Wiley and Sons, Inc., New York. 143 pages. $3.00.

Professor Pollard's book is a most welcome addition to the literature on Algebraic Number Theory in that it partially fulfills a long felt need for a read-

able English text on the subject. On the whole the book is well written and it is almost self-contained. The theorems are precisely stated and very complete cross references to earlier theorems make the proofs easy to follow. However, the definitions are occasionally not precisely stated and are often hidden in the midst of a discussion paragraph. (e.g. integral domain.) Efforts at motivation are frequent and usually successful. Practically the only prerequisites for the reader are a facility with the vocabulary of algebra and sufficient mathematical maturity to appreciate the problems raised.

The author begins by proving the unique factorization theorem of rational integers. He then raises the question as to whether the concept of integers can be generalized and if so, whether the factorization theorem continues to be valid. As examples he discusses the integers of $R(i)$, $R(\sqrt{-3})$, and $R(\sqrt{-5})$. In answering these questions he follows the classical approach using ideal theory, and the proofs are usually to be found in the works of Landau, Hecke, and Ore.

Chapters II through VII are devoted to introducing and developing such elementary facts of algebraic number theory as polynomial domains, field extensions, bases, discriminants, algebraic integers, units, primes, integral bases, and ideals. Also, the Liouville numbers are developed to demonstrate that not all complex numbers are algebraic. Throughout these chapters the reader is given a feeling for the subject by means of many fine examples taken from the quadratic and cyclotomic fields. However, the motivation of the definition of algebraic integers is weak and might be enhanced by the concepts of quotient field of rings and integral closure. Also, the author's aversion to the use of symmetric polynomials causes undue difficulties.

In Chapters VIII and IX the fundamental theorem on factorization of ideals is proved and a few of its consequences, such as the ramification of prime ideals, are studied. The unique factorization of integers into primes in an algebraic number field is shown to be equivalent to having every ideal in its ring of integers principal. Class numbers and Fermat's last problem are discussed in Chapter X. It is shown how parts of algebraic number theory were developed in an attempt to solve this famous problem. A theorem of Kummer is established, namely, if p is a regular odd prime, then $x^p+y^p=z^p$ has no solution in rational integers. Minkowski's theorem on lattice points and his theory of units comprise the final chapter. Unfortunately no examples are given in the last four chapters and as a consequence much of the theory remains sterile. For example, the theorems on ideals might have been considerably clarified by exhibiting the prime ideals of the quadratic fields.

All in all, one can be certain that the book will awaken in many of its readers an interest in this beautiful theory and thus the purpose of its authorship is fulfilled.

D. J. Lewis

Calculus and Analytic Geometry. By C. T. Holmes. McGraw-Hill, New York, 1950. 10+416 pp. $4.75.

The first five chapters of this book present the basic material of plane analytic geometry, the fundamentals of the calculus, both differential and integral, and applications, all involving polynomial functions. At least this much of the book is designed to be covered in one semester, either as a terminal course or as an introduction to the rest of the book.

Chapters 6–15, not all of which can be taken in one semester, include (in order) a discussion of curvilinear motion, the logarithmic, exponential, and trigonometric functions with applications, infinite series, differential equations solid analytic geometry, and the calculus for functions of several variables.

In the preface the author states that he makes free use of geometric intuition, preserving clear distinctions between what is proved and what is assumed; and he remarks that "we think that the book has as much rigor as most beginners can stand."

Some interesting questions are raised by a consideration of the structure of the book. Can a really satisfactory one-semester course in college mathematics for non-science students be devised, and, if so, what should be its subject matter? Is it possible that we underestimate the ability of beginners to absorb the punishment of rigor, and need rigor be punishing? There is no place in a short review for a discussion of questions of this sort, so that the author's premises with respect to the type of course for which he is writing will be accepted in what follows.

On the whole the book may be characterized as being conservative, with a number of novel features. The "point of division" formula, to give an example of conservatism, is presented in the unsymmetrical form, $x=(x_1+rx_2)/(1+r)$. On the other hand, the development of $\lim_{n\to\infty}(1+1/n)^n$ is straightforward and elegant, a heuristic discussion of tangents at the origin to the graph of $P(x, y)=0$ with constant term zero is excellent, and a modification of the second derivative test for extremes is really useful. The problems are well-chosen, carefully arranged, and contain instructive references and remarks.

An analysis of the author's style reveals at once his best feature and his greatest weakness. He has succeeded in transplanting the informality of a good class room teacher into the pages of a book. The text reads smoothly, points are brought up easily and naturally, the arguments are not labored. However, there is a lack of precision in some statements, which is at least annoying and may be confusing to the student. One would expect more care in the phrasing of statements which are formally labelled "Definition" or "Theorem" than is evidenced in the following example: "We say that $f(x)$ becomes infinite as x approaches a $\cdots$ if the value of $f(x)$ can be made to exceed any given number by taking x close enough to a." The fundamental theorem of the integral calculus is introduced by a consideration of area, but there is no mention of the

fact that the notion of area is as sophisticated as that of instantaneous speed, and that the area in question is *defined* as the limit of a sum. In 100 pages the reviewer encountered 9 instances of what he considers to be logical inaccuracies.

If more attention had been given to the phrasing of formal statements, this book would be an outstanding one of its kind.

R. A. ROSENBAUM

NEW BOOKS RECEIVED

Borderlands of Science. By Alfred Still. New York, Philosophical Library, Inc. ix+424 pp.

Quantum Mechanics. By Alfred Lande. New York, Pitman Publishing Company. x+307 pp. $5.50.

College Trigonometry. By William L. Hart. Boston, D. C. Heath and Company, 1951. viii+130 pp. $3.50.

Theory of Probability. By M. E. Munroe. New York, McGraw-Hill Book Company, 1951. viii+213 pp. $4.50.

Foundations of Analysis. By Edmund Landau. Chelsea Publishing Company, 1951. xiv+134 pp. No price given.

The Kernel Function and Conformal Mapping. Mathematical Surveys No. 5. By S. Bergman. American Mathematical Society, New York. vii+161 pp. No price given.

Algebra I and II. I—Die Grundlagen, viii+300 pp. DM 16. II—Die Theorie der Algebraischen Gleichungen, 3rd ed. viii+260 pp. DM 14. By O. Perron. Walter de Gruyter and Company, 1951.

Analyse Mathematique. Tome I, Analyse des Courbes, Surfaces et Fonctions Usuelles, Integrales Simples. ix+408 pp. 2000 fr. Port. 100 fr. Tome II, Equations differentielles, Developpements en series, Nombres complexes. Integrales multiples, Probabilities, Determinants, Exercises. 434 pp. 2200 fr. Port 110 fr. By Paul Appell. Paris, Gauthier-Villars, 1951.

Exercices de Mecanique. By H. Beghin, and G. Julia, Tome I, Fascicule 1. 2nd ed. Paris, Gauthier-Villars, 1946, vii+336 pp. No price given.

Problemes de Propagations guidees des Ondes Electromagnetiques, 2nd ed. By Louis de Broglie. Paris, Gauthier-Villars, 1951. vii+118 pp. $3.38.

Integraltafel, Zweiter Teil. Bestimmte Integrale, VI. By W. Grobner and N. Hofreiter. Springer-Verlag in Wien, 1950. $5.80.

Handbuch der Laplace-Transformation, Band 1. Theorie der Laplace-Transformation. By G. Doetsch. Basel, Verlag Birkhauser, 1950. 581 pp. Bound—78 fr.; unbound 74 fr.

Die Zweidimensionale Laplace-Transformation. By D. Voehlker and G. Doetsch. Basel, Verlag Birkhauser, 1950. 259 pp. Bound 43 fr.; unbound 35 fr.

The Main Street of Mathematics. By Edna E. Kramer. Oxford University Press, 1951. xii+321 pp. $5.00.

Tables d'Interets et D'Annuites. Editees Par le Credit Communal de Belgique, Brussels, 1950. 163 pp. No price given.

CLUBS AND ALLIED ACTIVITIES

EDITED BY L. F. OLLMANN, Hofstra College

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to L. F. Ollmann, Hofstra College, Hempstead, New York.

CLUB REPORTS, 1949–50

Pythagoreans, City College of the City of New York

The 1949–50 program of the mathematics club, *Pythagoreans*, of the City College of the City of New York included the following talks by undergraduates:

Concurrency and colinearity, by Milton Halem

Summation of series, by Jack Bricker

Basic considerations of topology, by Karl Brunell

Transformations, by Donald Newman of New York University

Partitions, by Bernard Weitzer

Elementary proof of a theorem by Hardy and Ramanujian, by Donald Newman of New York University

Integration in groups, by Carl Engelman

Extension of a function, by L. Rubel.

The following talks by members of the faculty were also given:

A problem in probability, by Prof. Henry Malan

The 4-color problem, by Prof. B. P. Gill

Cremona transformations, by Prof. Shelbourne Barber

Summation and integration, by Prof. W. H. Ingram.

The officers for 1950–51 include: President, Howard Young; Faculty Representative, Mr. Ingram.

Mathematics Society, Massachusetts Institute of Technology

The Mathematics Society of Massachusetts Institute of Technology held weekly meetings which featured lectures by various members of the faculty as well as by student members of the group. Among the lectures given were:

Exterior differential forms, by Prof. C. B. Allendoerfer

Finite difference equations and applications, by Prof. F. B. Hildebrand

The mathematics of Descartes, by Prof. D. J. Struik

Problems of topology, by Prof. G. W. Whitehead

The existence of continuous functions without derivatives, by Prof. W. Ambrose

Infinite systems of linear equations, by Dr. E. A. Coddington

What is dimension?, by Prof. W. Hurewicz

The isoperimetric problem, by Prof. I. S. Cohen

Calculus of variations, by Prof. G. B. Thomas

Boolean algebra as applied to the design of electrical switching circuits, by Mr. H. L. Reed, Jr.

Inversion of algebraic curves, by Mr. D. G. Aronson

Geometric inversion, by Mr. R. W. Preisendorfer.

The Society continued to publish its *Bulletin* with Mr. C. W. Bostick as editor.

The mathematics department exhibits at the M.I.T. Open House were constructed and staffed by the members of the Society under the supervision of Prof. R. D. Douglass and the Executive Committee of the Society.

The officers for the year were: President, D. G. Aronson; Vice-President, H. L. Reed, Jr.; Program Manager, R. Silver; Secretary-Treasurer, R. B. Kellogg.

Pi Mu Epsilon, Northwestern University

The following talks were given at the *Illinois Beta* Chapter of *Pi Mu Epsilon* meetings in 1949–50:

The development of logic, by Prof. Paul Henle of the Philosophy department

Why have cash?, by Prof. Jacob Marschak of the Cowles Commission, University of Chicago

A larger role for polar coordinates in analytics, by Prof. H. A. Simmons.

The annual initiation banquet was held in May, when 38 new members were initiated. Prof. E. H. C. Hildebrandt spoke on *The history and organization of Pi Mu Epsilon.*

The winners of the Pi Mu Epsilon mathematics contest held under the direction of Prof. Simmons were Richard Goldberg, Edmund Cohler, and Ann Evans, in that order.

Mathematics Society, The Cooper Union

The *Mathematics Society* of The Cooper Union heard papers on the following topics during 1949–50:

Matrices, by Irving Lowe

Nomographs, by Walter Kahn

Projective geometry, by Peter Redmond

Graphical solution of non-linear differential equations, by Harry Hochstadt

Group theory, by Mr. P. Chessin

Metric geometry, by Prof. J. N. Eastham

Imaginary curves, by Prof. F. H. Miller

Number theory, by Alan Berndt.

Officers for 1950–51 are: President, Peter J. Redmond; Vice-President and Treasurer, Irving J. Lowe; Secretary, Harry Schwarzlander; Faculty Adviser, Prof. J. N. Eastham.

Kappa Mu Epsilon, Mount Mary College

The *Wisconsin Alpha* chapter of *Kappa Mu Epsilon* held regular monthly meetings, 1949–50, at which the following talks were given by the members:

Non-Euclidean geometry, by Joan Daley

The fourth dimension, by Mary Kilkelly

History of the calculus, by Mary Hunt

Relativity, by Kathleen Hanley
Number systems, by Wanda Kropp
The slide rule, by Janet Haig
The planetarium, by Betty Prossen
Radar, by Bernadine Spitznogle
Aristotle, by Norma Harding.

Other activities included movies on the solar system, a Christmas party, a radio skit given by the initiates, and a dinner given for the initiates.

Two faculty members and two students attended the convention of the National Council of Mathematics Teachers held in Chicago. Miss Dorothy Karner, one of the students, read a paper in the Kappa Mu Epsilon section of the convention. Her paper was entitled *Geometric inversion* and was later printed in the Fall '50 issue of the *Pentagon*.

Officers for 1949–50 were: President, Dorothy Karner; Vice-President, Rosemary White; Secretary, Mary Hunt; Treasurer, Mary Kilkelly; Corresponding Secretary, Sister Mary Petronia; Faculty Sponsor, Sister Mary Felice.

Mathematics Club, Boston University

The Boston University *Mathematics Club* held its meetings twice a month, at which time various talks were given by the faculty and student members. Of the many discussions, two were outstanding. They were *Non-linear differential equations* and *Boolean algebra*.

Boston University was host to the annual meeting of the Greater Boston Intercollegiate Mathematics Club which was held in December, 1949. Prof. Elmer B. Mode, Chairman of the Mathematics Department, gave a humorous talk on *My mathematical scrapbook*. At this meeting Tufts College volunteered to be host in 1951.

A successful picnic was held in the Spring.

New officers for 1950–51 are: President, William Donovan; Vice-President, Barbara Torrey; Secretary, Elliot Croft; Treasurer, William Fitzgerald.

The Mathematics and Physics Club, College of St. Thomas

Papers presented to the *Mathematics and Physics Club* of the College of St. Thomas for 1949–50 included:

The brachistochrone problem and related topics, by E. J. Camp, Professor of Mathematics, Macalester College

Some early mathematical and astronomical tables, by Mr. W. D. Morgan, noted collector of early tables

A perpetual calendar, by Dr. Charles Hatfield, Jr., Professor of Mathematics, University of Minnesota.

Phenomena at the temperature of liquid helium, by Dr. J. R. Feldmeier, Professor of Physics

Some of the fallacies in mathematics, by Dr. W. S. Loud, Professor of Mathematics, University of Minnesota.

The *Mendel Forum Club* of St. Catherine's College attended the meeting at which Dr. Hatfield spoke, while the local club attended a meeting of the *Forum Club* at which Dr. J. F. Briggs spoke on *Your heart beat.*

The officers for the year 1950–51 are: President, Clarence B. Germain; Vice-President, R. P. Goblirsch; Secretary, Eugene Heath; Moderator, Dr. L. W. Sheridan.

Mathematics Society, University of Miami

The following talks were presented during the school year of 1949–50:
Hindu mathematics, by H. D. Sprinkle
Fallacies, by John Maecher
String models, by William Franzen
Magic squares, by Mabel Pauley
Computing machines, by Mr. John Kelley
Peano's space filling curve, by D. J. R. Foulis
Needle problem of Count Buffon, by D. J. R. Foulis
Integration of elliptic integrals, by Dr. H. F. MacNeish.

In addition to its regular meetings, the Mathematics Society held a Christmas Party and its Annual Spring Banquet.

The society has applied for membership in *Pi Mu Epsilon*, National Honorary Mathematics Fraternity, and has been accepted. The installation ceremonies are to be held in the Spring of 1951.

The officers for 1949–50 were: President, D. J. R. Foulis; Vice-President, John Maecher; Secretary, Mabel Pauley; Treasurer, Marjorie Stern (first semester) and Carolyn Palmer (second semester); Faculty Advisor, Prof. Georgia Del Franco.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

DIVISION OF MATHEMATICS OF THE NATIONAL RESEARCH COUNCIL

The National Research Council has announced the formation of a new division to be known as the Division of Mathematics. Professor Marston Morse of the Institute for Advanced Study has been appointed Chairman. Professor Marshall Stone, chairman of the Department of Mathematics of the University

of Chicago, will serve as Vice-Chairman. Professor J. R. Kline of the University of Pennsylvania has been appointed Executive Secretary. The new division will be composed of mathematicians who were formerly members of the Division of Mathematical and Physical Sciences, now the Division of Physical Sciences, and others to be appointed by the Chairman of the National Research Council.

PACIFIC JOURNAL OF MATHEMATICS

The Pacific Journal of Mathematics, a nonprofit corporation, announces a new journal of the same name, to be devoted to the publication of basic research articles in the various branches of mathematics. Publication of the *Pacific Journal of Mathematics* is sponsored by the following West Coast universities: the University of British Columbia; the California Institute of Technology; the University of California at Berkeley, Davis, Los Angeles and Santa Barbara; Oregon State College; the University of Oregon; the University of Southern California; Stanford University; Washington State College; and the University of Washington. The Institute for Numerical Analysis of the National Bureau of Standards is providing valuable clerical assistance and space for the *Journal*, as far as the managing editor is concerned. In addition, the American Mathematical Society is furnishing financial aid to the *Journal* during the initial stages of its existence.

A minimum of 600 pages per yearly volume of four issues is planned. The first issue of 160 pages is scheduled for March, 1951.

Subscription price for the *Journal* is $8.00 per year, with a reduction of $4.00 to individual faculty members of the above supporting institutions and to members of the American Mathematical Society. Subscriptions should be sent to the *Pacific Journal of Mathematics*, University of California Press, Berkeley 4, California.

Manuscripts for publication in the *Journal* should be submitted to any of the three editors: E. F. Beckenbach, University of California, Los Angeles 24, California; Herbert Busemann, University of Southern California, Los Angeles, California; R. M. Robinson, University of California, Berkeley 4, California.

SUMMER COURSES

The Department of Mathematics Education of the School of Education of New York University announces the following courses for its summer session: Dr. A. D. Bradley, teaching and curricular problems of college mathematics, applications of mathematics; Dr. J. J. Kinsella, research investigations in mathematics education.

The Statistical Laboratory of the University of California at Berkeley announces the following for the first summer session, June 18th to July 28th: Professor M. G. Kendall of the London School of Economics and Political Science, recent developments in the theory of statistics, seminar on time series and related problems; Professor Jerzy Neyman will be available for consultation on work leading to higher degrees.

PERSONAL ITEMS

Professor C. B. Allendoerfer of Haverford College has been appointed to a professorship at the University of Washington in Seattle where he will hold the position of Executive Officer of the Department of Mathematics. Professor R. M. Winger is retiring as Executive Officer but will continue on the active faculty. As was announced in an earlier issue Professor Allendoerfer will become Editor of the MONTHLY on January 1, 1952. After September 1, 1951 papers submitted for publication in the MONTHLY should be sent directly to Professor C. B. Allendoerfer, Department of Mathematics, University of Washington, Seattle 5, Washington and not to the present editor, Dr. C. V. Newsom.

Professor H. A. Robinson of Agnes Scott College was the representative of the Association at the inauguration of President O. C. Aderhold of the University of Georgia on May 9, 1951.

Professor Emeritus Mary E. Sinclair of Oberlin College represented the Association at the inauguration of President A. S. Knowles of the University of Toledo on May 9, 1951.

Kent State University reports that a Mathematics Workshop for Secondary School Teachers was conducted by its Mathematics Department on April 14, 1951. The following members of the Department served on the Workshop Committee: Emalou Brumfield as chairman, Carl Lowry and E. T. Stapleford.

University of Rochester announces: Dr. Eduardo Caianiello, formerly of the University of Naples, has been appointed to an assistant professorship; Mr. David Barton, formerly instructor at Roberts College, has been appointed Graduate Instructor; Mr. Donald Smith, previously of the University of Rochester, Mr. Charles Younger, a graduate of West Texas Teachers College, and Mr. Deiter Gaier of the Technische Hochschule at Stuttgart have received appointments as teaching assistants; Assistant Professor Dorothy L. Bernstein has a year's leave of absence which she is spending at the Institute for Advanced Study; Instructor Trevor McMinn is now a graduate student at the University of California at Berkeley; Mr. Warren Stenberg who taught at the University during the first semester is now a graduate assistant at the University of California at Berkeley; Teaching Assistant Eugene Trabka has accepted a position at the Cornell Aeronautical Laboratory, Buffalo, New York.

Miss Janet E. Abbey, formerly an instructor at the University of Buffalo, has been appointed Head of the Department of Mathematics of Griffith Institute and Central School, Springville, New York.

Mr. A. D. Anderson has received an appointment as mathematician in the Naval Research Laboratory, Washington, D. C.

Professor H. C. Ayres of Jersey City Junior College is now in military service.

Assistant Professor W. H. Badgley, Jr., of Florence State Teachers College, Alabama, is a graduate student at Vanderbilt University.

Miss Jean M. Baldwin, formerly a student at Carleton College, has been appointed to a graduate assistantship at the University of Oklahoma.

Assistant Professor W. E. Barnes of the College of William and Mary has accepted a position as mathematician at the United States Naval Proving Ground, Dahlgren, Virginia.

Dr. F. A. Beeler, previously a graduate student at the University of Michigan, has been appointed to a professorship at Western Michigan College of Education.

Mr. P. R. Beesack, formerly a student at McMaster University, has been appointed to a graduate assistantship at Washington University.

Mr. D. C. Benson, a student at Pomona College, has been appointed to a teaching assistantship at Stanford University.

Mr. C. R. Bonnell of the School of Mines and Metallurgy of the University of Missouri has been appointed Research Engineer at the Minneapolis-Honeywell Company, Minneapolis, Minnesota.

Mr. L. F. Boron of the University of Kentucky has a position as mathematician in the Navy Department, Washington, D. C.

Mr. J. E. Brown, previously a student at the University of Georgia, has been appointed acting instructor at Florida State University.

Mr. R. A. Burrows, formerly a student at Albion College, is employed by General Motors Corporation, Detroit, Michigan.

Mr. L. G. Campbell of New Jersey State Teachers College has been appointed to an instructorship at Teachers College, Columbia University.

Associate Professor L. Virginia Carlton of Northwestern State College has been appointed Head of the Department of Mathematics of Wesleyan College.

Mr. H. C. Carter of David Lipscomb College is now in military service.

Mr. B. B. Clark of Grinnell College is in the United States Air Force.

Mr. M. J. Cleveland, part-time instructor at the University of Florida, is in the United States Navy.

Mr. J. J. Fischer of Morgan Park Junior College has accepted a position as research engineer in the Aerophysics Laboratory, North American Aviation, Incorporated, Downey, California.

Mr. H. C. Griffith, formerly assistant instructor at the University of Missouri, has been appointed to a teaching assistantship at the University of Tennessee.

Mr. Ralph Hafner of the University of Dayton has a position as mathematician in the Naval Ordnance Plant, Indianapolis, Indiana.

Mr. W. C. Hobbs of Hampton Institute has been appointed Head of the Department of Mathematics of Morris Brown College.

Mr. Julius Honig, previously research assistant at Long Island College of Medicine, has accepted a position as mathematician with the Raytheon Electric Corporation, Waltham, Massachusetts.

Assistant Professor S. T. Hu of Tulane University has been promoted to an associate professorship; he will be on leave of absence during the year 1951—52 and will spend this year at the Institute for Advanced Study.

Dr. H. D. Huskey, chief of the Machine Development Unit, Institute for

Numerical Analysis, National Bureau of Standards, has been promoted to the position of Assistant Director.

Mr. T. C. Hutchison, who has been an electrical engineer at Haller, Raymond and Brown, Incorporated, State College, Pennsylvania, has accepted a position as project engineer at the Sperry Gyroscope Company, Great Neck, New York.

Mr. R. G. Ingle of Sandia Corporation, Albuquerque, New Mexico, has a position as vibration engineer at Chance Vought Aircraft, Dallas, Texas.

Mr. O. C. Juelich, formerly a student at Hofstra College, has been appointed to an assistantship at Ohio State University.

Professor Mark Kac of Cornell University will serve as visiting lecturer at Massachusetts Institute of Technology during the first part of the summer session.

Professor Emeritus Edward Kasner of Columbia University was a teacher in the New School for Social Research during the spring term.

Mr. C. E. Kerr of LaSalle College has been appointed to a graduate assistantship at the University of Delaware.

Mr. W. S. Knight, previously a student at the University of Georgia, is now in the United States Army.

Mr. B. W. Marks of Midland High School, Midland, Texas, is in the United States Army.

Dr. W. H. Marlow, who has been a graduate assistant at the State University of Iowa, has a position as research associate, Logistics Research Projects, George Washington University.

Professor Karl Menger of Illinois Institute of Technology delivered a series of lectures on metric geometry at the Sorbonne, University of Paris, during April-May, 1951.

Instructor R. H. Oehmke of the University of Detroit is now a graduate student at the University of Chicago.

Instructor F. R. Olson, Kent State University, has been appointed to a part-time instructorship at Duke University.

D. H. A. Palmer, Head of the Physical Science Department of Midwestern University, is now a chemist in the Research Department of the United Gas Corporation, Shreveport, Louisiana.

Mr. J. D. Riley, previously a mathematician at the Naval Research Laboratory, is a graduate student at the University of Kansas.

Auxiliary Professor Margarita Rodriguez of Instituto del Vedado is now a mathematics teacher at Vedado High School, Vedado, Havana.

Dean Evelyn C. Rusk of Wells College will resign from her position as dean on July 1, 1951; after a year's leave of absence she will return to the college as professor of mathematics.

Mr. J. D. Rutledge, previously a student at Swarthmore College, is now a Programmer Trainee at Eckert-Mauchly Computer Corporation, Philadelphia, Pennsylvania.

Mr. L. R. Schlauch, who has been a graduate student at the University of Virginia, has received an appointment as analyst for the Department of Defense, Armed Forces Security Agency, Washington, D. C.

Mr. R. J. Semple has been appointed teaching fellow at the University of Toronto.

Colonel W. E. Sewell is Professor of Military Science and Tactics at the State University of Iowa.

Sister M. Laurine of Mount St. Joseph Junior College has been appointed Registrar and Instructor at Brescia College, Owensboro, Kentucky.

Graduate Assistant O. D. Smith of Oregon State College is now a teaching assistant at the University of Southern California.

Mr. B. R. Snyder, formerly of Johns Hopkins University, has accepted a position with the law firm of K. F. Steinmann, Baltimore, Maryland.

Graduate Assistant J. C. Sorenson of the University of Oregon has been appointed to a graduate assistantship at Utah State Agricultural College.

Mr. S. P. Spaulding, formerly a student at Boston University, has received an appointment as ordnance engineer at Naval Torpedo Station, Newport, Rhode Island.

Dr. M. D. Springer has accepted a position as mathematical statistician with United States Naval Ordnance, Indianapolis, Indiana.

Mr. J. A. Standerfer, graduate assistant at North Texas State College, has been called to active duty in the United States Navy.

Mr. G. W. Starch, formerly a student at Oklahoma Agricultural and Mechanical College, has a position as junior mining geologist, New Cornelia Mine, Phelps Dodge Corporation, Ajo, Arizona.

Associate Professor J. K. Sterrett of Marshall College is now a mathematician at Ballistics Research Laboratory, Aberdeen Proving Ground, Maryland.

Mr. W. B. Stovall, Jr., statistician at the Bureau of Vital Statistics, State Board of Health, Jacksonville, Florida, is serving in the United States Navy.

Associate Professor H. L. Turrittin of the University of Minnesota has been appointed to an associate professorship at Princeton University.

Associate Professor W. R. Van Voorhis is on leave of absence from Fenn College and is serving as Director of Training for the Military and Civil Defense Commission of Pennsylvania.

Dr. S. S. Walters, who has been a mathematician at the Rand Corporation, has been appointed to an instructorship at the University of California at Los Angeles.

Mr. Chih-Yi Wang has been appointed to a teaching assistantship at the University of Minnesota.

Instructor E. H. Wang of the University of Cincinnati has established a Mathematical Engineering Service in Cincinnati.

Mr. A. W. Wortham, instructor and research assistant at Oklahoma Agricultural and Mechanical College, has a position as senior project analytical engineer in Dynamics Analysis Section, Chance Vought Aircraft, Dallas, Texas.

Assistant Professor G. C. Zader of The Citadel is on leave of absence and is serving in the United States Navy.

Mr. L. H. Cutting of Kansas City High School died on February 13, 1951. He had been a member of the Association for thirty years.

Professor Emeritus R. M. McDill of Hastings College died on March 19, 1951. He had been a member of the Association for thirty years.

Dean Emeritus H. L. Slobin of the University of New Hampshire died February 22, 1951. He was a charter member of the Association.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following eighty-four persons have been elected to membership by the Board of Governors on applications duly certified:

J. C. ABBOTT, Ph.D.(Notre Dame) Asso. Professor, U. S. Naval Academy, Annapolis, Md.

J. T. AHLIN, M.A.(Southern California) Research Engineer, Douglas Aircraft Company, Santa Monica, Calif.

W. A. AL-SALAM, B.S.(California) Grad. Student, University of California, Berkeley, Calif.

W. E. ANDRUS, JR., A.B.(Syracuse) Technical Engineer, International Business Machines Corporation, Endicott, N. Y.

MRS. EDITH W. ANNIS, A.B.(Hunter) Chairman, Department of Mathematics, Mamaroneck Senior High School, N. Y.

E. S. ASHCRAFT, M.A.(North Carolina) Asst. Professor, Stetson University, DeLand, Fla.

A. V. BANES, M.S.(Oklahoma A and M) Asst. Professor, Trinity University, San Antonio, Tex.

J. H. BANKS, Ph.D.(George Peabody) Asso. Professor, George Peabody College for Teachers, Nashville, Tenn.

T. J. BENAC, Ph.D.(Yale) Asso. Professor, U. S. Naval Academy, Annapolis, Md.

J. W. BLATTNER, Student, Central College, Fayette, Mo.

R. G. BROWN, M.A.(Yale) Research Engineer, Willow Run Research Center of the University of Michigan, Ypsilanti, Mich.

J. P. CARBERRY, B.S.(Roanoke) Grad. Student, Carnegie Institute of Technology, Pittsburgh, Pa.

A. J. CARLAN, B. A.(Brooklyn C.) 214 Avenue N, Brooklyn, N. Y.

R. J. CARY, Student, Harpur College, Endicott, N. Y.

Y. W. CHEN, Dr.(Göttingen) Asso. Professor, University of Oklahoma, Norman, Okla.

K. J. COHEN, Student, Reed College, Portland, Ore.

J. I. DERR, B.S.(Southern Methodist) Grad. Student, Southern Methodist University, Dallas, Tex.

ANGELO DIBELLA, B.S.(New Mexico A and M) Research Assistant, Sandia Corporation, Albuquerque, N. M.

R. E. DOWDS, Student, Kent State University, Ohio

MYRTLE EDWARDS, M.A.(Georgia) Teacher, Mars Hill College, N. C.

G. E. FORSYTHE, Ph.D.(Brown) Mathematician, National Bureau of Standards, Los Angeles, Calif.

BARBARA A. FUTRAL, Student, Agnes Scott College, Decatur, Ga.

JOAQUIN GALVAN P., Student, St. Mary's University, San Antonio, Tex.

G. H. GLEISSNER, M.A.(Columbia) Mathematician, U. S. Naval Proving Ground, Dahlgren, Va.

F. S. GOEPPER, JR., M.A. (Cincinnati) Grad. Student, Harvard University, Cambridge, Mass.

A. J. GOLDMAN, Student, Brooklyn College, N. Y.

LOUISA S. GRINSTEIN, B.A. (Buffalo) Grad. Student, University of Buffalo, N. Y.

M. P. GUHSE, B.S. (Hamilton) Grad. Student University of Massachusetts, Amherst, Mass.

ROBERT HARTRANFT, B.S. (Stanford) Mathematician, Naval Air Missile Test Center, Pt. Mugu, Calif.

E. Y. HILL, Student, University of British Columbia, Vancouver, B. C. Canada.

MR. JIRO ISHIHARA, M. S. (Northwestern) Grad. Student, Northwestern University, Evanston, Ill.

P. F. IVERSON, A.B. (Hastings) Head Department of Mathematics, Potomac State School of West Virginia University, Keyser, W. Va.

M. W. JONES, Ph.D. (Colorado) Asso. Professor, Adams State College, Alamosa, Colo.

T. L. JORDAN, JR., M.A. (Vanderbilt) Teaching Fellow, Vanderbilt University, Nashville, Tenn.

T. K. KAWATA, B.A. (Southwestern C.) Grad. Student, DePaul University, Chicago, Ill.

MRS. HELEN S. KEEFER, Student, Iowa Wesleyan College, Mount Pleasant, Iowa

S. R. KNOX, M.A. (Mississippi) Asst. Professor, Millsaps College, Jackson, Miss.

I. H. KRAL, B.S.R.E. (Indiana Tech.) Junior Engineer, Bendix Aviation Research Laboratories, Detroit, Mich.

R. A. C. LANE, A.B. (Indiana) Grad. Assistant, Lehigh University, Bethlehem, Pa.

L. I. LOWELL, B.S.M.E. (Michigan State C.) Grad. Student, Michigan State College, East Lansing, Mich.

W. A. LUCAS, M.E. (Stevens Tech.) Faculty Instructor, Stevens Institute of Technology, Hoboken, N. J.

M. T. MACNEIL, Student, University of Detroit, Mich.

J. E. MADDEN, M.S. (Brown) Instructor, Portsmouth Priory School, Rhode Island

L. E. MAHURON, B.A. (Hardin-Simmons) Supervisor, I. B. M. Section, Sandia Corporation, Albuquerque, N. M.

P. L. MARSHALL, Student, Hofstra College, Hempstead, N. Y.

J. E. MAXFIELD, M.S. (Wisconsin) Instructor, University of Oregon, Eugene, Ore.

AUDREY L. MICHAELS, Student, Brooklyn College, N. Y.

E. R. MICHALIK, M.A. (Pittsburgh) Asst. Professor, University of Pittsburgh, Pa.

R. W. MOLLER, Ph.D. (Catholic U.) Instructor, Catholic University of America, Washington, D. C.

C. B. MORREY, JR., Ph.D. (Harvard) Professor, University of California, Berkeley, Calif.

C. R. MORRIS, M.A. (Kentucky) Mathematician, U. S. Navy Hydrographic Office, Suitland, Md.

G. R. MOTT, Student, Hofstra College, Hempstead, N. Y.

MARY L. MURPHY, M.S. (Boston C.) Secretary, Mathematics Department, Boston College, Chestnut Hill, Mass.

THEODORA S. NELSON, M.S. (Illinois) Asst. Professor, Nebraska State Teacher's College, Kearney, Neb.

C. D. PARKER, Student, Michigan State College, East Lansing, Mich.

R. E. PINGRY, Ph.D. (Cornell) Asst. Professor, University of Illinois, Urbana, Ill.

LILY B. POWELL, M.A. (Colorado) Instructor, University of Colorado, Boulder, Colo.

J. B. PRYZIE, B.S. (C.C.N.Y.) Consolidated Edison Company, New York, N. Y.

V. ELISE QUALLS, A.B. (Berea) Instructor, Tennessee Polytechnic Institute, Cookeville, Tenn.

B. D. RAMSDELL, M.S. (Michigan) Instructor, Orlando Junior College, Fla.

R. M. REDHEFFER, Ph.D. (M.I.T.) Instructory, University of California at Los Angeles, Calif.

H. L. RICE, Student, St. Mary's University, San Antonio, Tex.

J. W. RIDLEY, JR., Junior Chemist, Allied Chemical and Dye Corporation, Philadelphia, Pa.

P. D. RITGER, A.M. (Pennsylvania) Grad. Student, New York University, N. Y.

T. J. RIVLIN, M.A. (Harvard) Grad. Student, Harvard University, Cambridge, Mass.

F. M. ROLAND, Student, St. Mary's University, San Antonio, Tex.

MRS. ANNIE G. H. SASSER, M.S. (Prairie View A. and M.) Asst. Professor, Tennessee Agricultural and Industrial State College, Nashville, Tenn.

HERBERT SCARF, Student, Temple University, Philadelphia, Pa.

SISTER CHARLES ANNE, M.A. (Ohio State) Dean of Studies, College of St. Mary of the Springs, Columbus, Ohio

GUSTAVE SOLOMON, Student, Yeshiva College, New York, N. Y.

MARVIN STERN, M.S. (N.Y.U.) Mathematics Consultant, Republic Aviation Corporation, Farmingdale, N. Y.

J. A. SULLIVAN, Ph.D. (Indiana) Instructor, University of Notre Dame, Ind.

ANNA L. TAYLOR, A.B. (Southwest Missouri) Asst. Instructor, University of Missouri, Columbia, Mo.

R. J. TEED, M.A. (Illinois) Instructor, Fournier Institute of Technology, Lemont, Ill.

W. E. TIMON, JR., B.S. (Northwestern S. C. of La.) Grad. Fellow, Tulane University, New Orleans, La.

MRS. HALSEY K. TRUAN, B.S. (Pittsburgh) Laboratory Technician, Atomic Power Division, Westinghouse Electric Corporation, Pittsburgh, Pa.

B. O. VANHOOK, M.A. (Vanderbilt) Head, Department of Mathematics, Mississippi Southern College, Hattiesburg, Miss.

E. H. WEISS, Sc.M. (Brown) Mathematician, U. S. Naval Proving Ground, Dahlgren, Va.

H. E. WEISSLER, B.Sc. (St. Mary's) Instructor, St. Mary's University, San Antonio, Tex.

MARGARET M. WELCH, Student, Texas Christian University, Ft. Worth, Tex.

NATHAN WETROGAN, B.A. (Brooklyn C.) Grad. Student, Brooklyn College, N. Y.

LOIS B. WHITMAN, B.A. (U.C.L.A.) Teacher, Fairfax High School, Los Angeles, Calif.

R. G. WOODLE, JR., B.S. (Drury) Instructor, Missouri School of Mines and Metallurgy, Rolla, Mo.

B. K. YOUSE, B.S. (Alabama Poly.) Grad. Assistant, University of Georgia, Athens, Ga.

CALENDAR OF FUTURE MEETINGS

Joint meeting with American Society for Engineering Education, Michigan State College, East Lansing, June 25–26, 1951.

Thirty-second Summer Meeting, University of Minnesota, Minneapolis, September 3–4, 1951.

Thirty-fifth Annual Meeting, Brown University, Providence, Rhode Island, December 29, 1951.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN

ILLINOIS

INDIANA

IOWA

KANSAS

KENTUCKY

LOUISIANA-MISSISSIPPI, Northwestern State College, Natchitoches, Louisiana, February 15–16, 1952.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA

METROPOLITAN NEW YORK

MICHIGAN

MINNESOTA, North Dakota Agricultural College, Fargo, October 6, 1951.

MISSOURI

NEBRASKA

NORTHERN CALIFORNIA, University of California, Berkeley, January 26, 1952.

OHIO

OKLAHOMA

PACIFIC NORTHWEST, State College of Washington, Pullman, June 15, 1951.

PHILADELPHIA, University of Pennsylvania, Philadelphia, November 24, 1951.

ROCKY MOUNTAIN

SOUTHEASTERN, Georgia Institute of Technology and Agnes Scott College, Atlanta, March 21–22, 1952.

SOUTHERN CALIFORNIA, Occidental College, Los Angeles, March 8, 1952.

SOUTHWESTERN, University of Arizona, Tucson, April 11–12, 1952.

TEXAS

UPPER NEW YORK STATE

WISCONSIN

Calculus

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CONTENTS

George Abram Miller . . . . . H. R. Brahana 447
The Story of Tangents . . . . . J. L. Coolidge 449
A New Absolute Geometric Constant? . . . Robin Robinson 462
Hyperbolic Trigonometry Derived from the Poincaré Model . . . . . Howard Eves and V. E. Hoggatt, Jr. 469
The Beta Function . . . . . C. S. Ogilvy 475
The William Lowell Putnam Mathematical Competition L. E. Bush 479
Mathematical Notes . . . . H. S. Thurston, F. A. Valentine 483
Classroom Notes . . . . . R. S. Underwood, M. R. Spiegel 487
Elementary Problems and Solutions 491
Advanced Problems and Solutions 495
Recent Publications 501
Clubs and Allied Activities 505
News and Notices 507
Mathematical Association of America 513
New Members 513
New Sectional Governors of the Association 515
Joint Meeting of the Association with A.S.E.E. 515
March Meeting of the Southern California Section 517
March Meeting of the Southwestern Section 519
Calendar of Future Meetings 522

AUGUST-SEPTEMBER 1951